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CONTENTS

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AN INTEGRO-DIFFERENTIAL BOUNDARY VALUE PROBLEM.*

By William T. Reid.

1. **Introduction.** Lichtenstein [12][1] has treated by means of the theory of quadratic forms in infinitely many variables a boundary value problem involving a single integro-differential equation of the second order and a special set of two-point boundary conditions. Under certain conditions he proved the existence of infinitely many characteristic values, and also established an expansion theorem for functions in terms of the corresponding characteristic solutions. More recently, Lichtenstein [13] has used the results of his previous paper to prove by expansion methods sufficient conditions for a weak relative minimum in the isoperimetric problem of the calculus of variations. Courant ([6], Sections 5 and 13) has treated by the method of difference equations an integro-differential boundary problem similar to that considered by Lichtenstein.

Tamarkin [21][2] has developed in two papers a somewhat general theory for an integro-differential boundary problem consisting of a single linear integro-differential equation of the n-th order and boundary conditions which involve not only the end-values of the solution and its first $n-1$ derivatives at two points, but also involve integral terms containing the solution functions. The outstanding feature of Tamarkin's treatment is the repeated use of the notion of a Green's function for the integro-differential system. In an unpublished dissertation Jonah [11] has proved the existence of a Green's matrix for a boundary problem associated with a system of integro-differential equations of the first order, and has extended the principal results of Tamarkin's papers to such a system.

The boundary problem which is here considered may be formulated as follows. Let

$$(1.1)\qquad I[\eta]=2Q[\eta(a),\eta(b)]+\int_a^b 2\omega(x,\eta,\eta')dx + \int_a^b\int_a^b M_{ij}(x,t)\eta_i(x)\eta_j(t)\,dx\,dt,$$

* The present paper contains the revised form of results presented to the American Mathematical Society under the titles "*An integro-differential boundary problem,*" December 28, 1934 and "*A boundary value problem associated with the calculus of variations,* II," April 19, 1935. Received by the Editors, September 29, 1937.

[1] Numerals in square brackets refer to the bibliography at the end of the present paper.

[2] Other references to literature on integro-differential boundary problems will be found in the introduction of Tamarkin's first paper.

where $\eta \equiv [\eta_i(x)]$ $(i = 1, \cdots, n)$ and $\eta_i(x)$ are real-single-valued functions on $a \leq x \leq b$, ω and Q are quadratic forms in the $2n$ variables η_i, η'_i and $\eta_i(a)$, $\eta_i(b)$, respectively. This paper treats the system consisting of the Euler-Lagrange integro-differential equations and transversality conditions for the problem of minimizing $I[\eta]$ in a class of arcs

$$(1.2) \qquad \eta_i = \eta_i(x) \qquad (i = 1, \cdots, n;\ a \leq x \leq b)$$

which satisfy a set of ordinary linear differential equations of the first order

$$(1.3) \quad \Phi_\alpha(x, \eta, \eta') \equiv \Phi_{\alpha\pi_i}(x)\eta'_i + \Phi_{\alpha\eta_i}(x)\eta_i = 0 \quad (\alpha = 1, \cdots, m < n),$$

together with linear homogeneous end-conditions

$$(1.4)\ \Psi_\gamma[\eta(a), \eta(b)] \equiv \Psi_{\gamma;ia}\eta_i(a) + \Psi_{\gamma;ib}\eta_i(b) = 0\ (\gamma = 1, \cdots, p \leq 2n),$$

and which are such that

$$(1.5) \qquad K[\eta] \equiv \int_a^b (\eta_1^2 + \cdots + \eta_n^2)dx = 1.$$

In the particular case when $M_{ij}(x, t) \equiv 0$ the expression (1. 1) is of the form of the second variation of a problem of Bolza in the calculus of variations and the above described boundary problem is the so-called accessory boundary problem, and has been treated by various authors (see Morse [15] and [16], Reid [18] and [19], Hu [9], Hölder [8], Birkhoff and Hestenes [1], and Wiggin [22]). In the general case, expression (1. 1) is the form of the second variation of a more general problem of the calculus of variations. The boundary value problem described above includes as a very special case the problem considered by Lichtenstein. It also includes a class of problems associated with a single self-adjoint linear integro-differential equation of even order. It is to be noted, however, that the boundary conditions of this problem are two-point conditions, and hence are not of as general a character as those of the problem treated by Tamarkin.

The hypotheses upon which the analysis is based are stated in Section 2 and in Section 3 some properties of the boundary problem are discussed. In Section 4 there are defined successive classes S_s $(s = 1, 2, \cdots)$ of arcs η in which we consider the problem of minimizing $I[\eta]$, and it is shown that the greatest lower bound of $I[\eta]$ in S_s is a characteristic value of our problem. The method of proof is similar to that previously used by the author [18] for the differential problem to which the above problem reduces whenever

$M_{ij}(x, t) \equiv 0$. Use is now made of a Green's matrix for an integro-differential system as introduced by Tamarkin and Jonah.

Sections 5 and 6 are devoted, respectively, to comparison and oscillation theorems. These theorems are generalizations of the Sturmian comparison and oscillation theorems for a single second-order linear differential equation (see, for example, Ince [10], Chapter X). These theorems have been previously established for a problem of the above sort involving differential equations, that is, for which $M_{ij}(x, t) \equiv 0$, and which satisfies certain additional normality assumptions.[3] *The present paper gives for the first time such theorems not involving any assumptions of normality on sub-intervals. The definitions of focal and conjugate points given in Section* 6 *are also new.* It is to be remarked that for an integro-differential system of the sort here considered, even though it be identically normal, one can not in general define the focal points as zeros of a certain determinant. In fact, so far as the author knows, comparison and oscillation theorems have not been previously given for even the simplest type of integro-differential system here considered which does not reduce to a differential system, that is, for which $M_{ij}(x, t) \not\equiv 0$.

In Section 7 we consider an integro-differential problem which is in general non-linear in the parameter. There are obtained theorems on the existence of real characteristic values, as well as comparison and oscillation theorems, again without any assumption of normality on sub-intervals. Of extreme significance is the method of proof used. There is associated with the given problem an auxiliary problem which is linear in a second characteristic parameter, and the characteristic values of the new problem are considered as functions of the original parameter. Comparison and oscillation theorems are immediate consequences of the corresponding theorems for the associated problem and, as proved in Sections 5 and 6, for this latter problem these theorems follow from the extremizing properties of the characteristic values. This method of making the theory of a problem non-linear in the parameter depend upon the corresponding theory for a problem linear in a second parameter seems to be highly significant, in view of the fact that problems which are linear in the parameter lend themselves readily to treatment by diverse methods.[4] As far as the author

[3] See Morse [14], [15], [16], and Hu [9]. Morse [16] has stated his results for a differential problem which involves no auxiliary differential equations $\Phi_a = 0$, and which satisfies the hypothesis of Theorem 7 of the present paper. As he points out, if suitable normality assumptions are made, his methods extend immediately to the more general differential boundary value problem. Birkhoff and Hestenes [1] have obtained these results for a differential system involving no auxiliary differential equations $\Phi_a = 0$.

[4] By a similar treatment one may readily extend some of the important results of

knows, however, the method has not been used previously for even the simplest problem of the type here treated, even in the case of a self-adjoint differential equation of the second order with separated end-conditions which satisfies the conditions prescribed in the Sturmian theory.

Finally, in Section 8 there is discussed briefly the idea of a Green's matrix for a system of integro-differential equations of the first order together with a set of two-point boundary conditions, and the fundamental theorems concerning a problem and its adjoint problem are given.

2. Notation and preliminary remarks. Throughout the first seven sections of this paper the following subscripts have the ranges indicated: $i, j, k = 1, \cdots, n$; $\alpha, \beta = 1, \cdots, m$; $\sigma, \tau = 1, \cdots, 2n$; $\gamma, \nu = 1, \cdots, p$; $\theta, \phi = 1, \cdots, 2n - p$. The repetition of a subscript in a single term of an expression will denote summation with respect to that subscript over its range of definition. Partial derivatives of $\omega(x, \eta, \pi)$, $\Phi_\alpha(x, \eta, \pi)$ with respect to the variables η_i, π_i will be denoted by writing these variables as subscripts; correspondingly, derivatives of Q and Ψ_γ with respect to the arguments $\eta_i(a)$, $\eta_i(b)$ will be denoted by Q_{ia}, $\Psi_{\gamma;ia}$, Q_{ib}, $\Psi_{\gamma;ib}$, respectively.

The analysis of the paper is based upon the following hypotheses:

(H_1) The coefficients of the quadratic form $\omega(x, \eta, \pi)$ and the linear expressions $\Phi_\alpha(x, \eta, \pi)$ are real-single-valued functions of x on ab. The functions $\omega_{\pi_i \pi_j}$, $\omega_{\pi_i \eta_j}$, $\Phi_{\alpha \pi_j}$ are of class C^1, and the functions $\omega_{\eta_i \eta_j}$, $\Phi_{\alpha \eta_j}$ are continuous on this interval. The functions $M_{ij}(x, t)$ are continuous on $a \leq x$, $t \leq b$, and $M_{ij}(x, t) = M_{ji}(t, x)$. Finally, the matrix $\| \Phi_{\alpha \pi_j} \|$ is of rank m on ab, the coefficients of the quadratic form Q and the linear homogeneous expressions Ψ_γ are real constants, and the matrix $\| \Psi_{\gamma;ia} \, \Psi_{\gamma;ib} \|$ has rank p.

An arc $\eta \equiv [\eta_i(x)]$ will be called *differentially admissible* if the functions $\eta_i(x)$ are of class D^1 on ab, and satisfy the equations $\Phi_\alpha = 0$ on this interval. An arc whose end values at a and b satisfy $\Psi_\gamma = 0$ will be said to be *terminally admissible*. Finally, an arc which is both differentially and terminally admissible will be called *admissible*.

(H_2) The quadratic form $\omega_{\pi_i \pi_j}(x) u_i u_j$ is positive for values $(u_i) \neq (0_i)$ and satisfying $\Phi_{\alpha \pi_j}(x) u_j = 0$ $(\alpha = 1, \cdots, m)$.

This hypothesis implies, in particular, that the matrix

$$(2.1) \qquad \left\| \begin{matrix} \omega_{\pi_i \pi_j} & \Phi_{\alpha \pi_i} \\ \Phi_{\beta \pi_j} & 0_{\beta \alpha} \end{matrix} \right\| \qquad (i, j = 1, \cdots, n;\ \alpha, \beta = 1, \cdots, m)$$

is non-singular on ab.

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the Hilbert-Schmidt theory of linear equations to corresponding integral equations involving the parameter non-linearly.

(H_3) There exist p differentially admissible arcs $\eta_i = \eta_{i\nu}(x)$ $(\nu = 1, \cdots, p)$ such that the determinant $|\Psi_\gamma[\eta_\nu(a), \eta_\nu(b)]|$ is different from zero.

Hypothesis (H_3) is a condition of normality with respect to the differential equations (1.3) and the end-conditions (1.4). It implies, in particular, that the conditions Ψ_γ are linearly independent. One may show that if η is a minimizing arc for the problem of the calculus of variations defined in Section 1, and satisfying the above hypotheses, then there exists a constant λ and functions $\mu_a(x)$ such that if we define

$$(2.2) \qquad \Omega(x, \eta, \pi, \mu) = \omega(x, \eta, \pi) + \mu_a \Phi_a(x, \eta, \pi),$$

$$(2.3) \qquad J_i(\eta, \mu) = (d/dx)\Omega_{\pi_i}(x, \eta, \eta', \mu) - \Omega_{\eta_i}(x, \eta, \eta', \mu) - \int_a^b M_{ij}(x, t)\eta_j(t)\,dt,$$

then the set η, μ, λ satisfies the integro-differential system

$$(2.4) \qquad J_i(\eta, \mu) + \lambda\eta_i = 0, \qquad \Phi_a(x, \eta, \eta') = 0;$$

moreover, there exist constants d_γ satisfying with the end values of η_i, μ_a the end conditions

$$(2.5) \qquad \begin{aligned} &Q_{ia}[\eta] + d_\gamma \Psi_{\gamma;\, ia} - \Omega_{\pi_i}(x, \eta, \eta', \mu)|^{x=a} = 0, \\ &Q_{ib}[\eta] + d_\gamma \Psi_{\gamma;\, ib} + {}_{ib} + \Omega_{\pi_i}(x, \eta, \eta', \mu)|^{x=b} = 0, \\ &\Psi_\gamma[\eta(a), \eta(b)] = 0. \end{aligned}$$

Since the matrix (2.1) is non-singular, the set of $m + n$ equations

$$(2.6) \quad \zeta_i = \Omega_{\pi_i}(x, \eta, \pi, \mu), \ \Phi_a(x, \eta, \pi) = 0 \quad (\alpha = 1, \cdots, m;\ i = 1, \cdots, n)$$

has unique solutions

$$(2.7) \qquad \pi_i = A_{ij}(x)\eta_j + B_{ij}(x)\zeta_j, \qquad \mu_a = l_{aj}(x)\eta_j + m_{aj}(x)\zeta_j.$$

When these values are substituted in $\Omega_{\eta_i}(x, \eta, \pi, \mu)$ we obtain

$$(2.8) \qquad \Omega_{\eta_i}(x, \eta, \pi, \mu) = C_{ij}(x)\eta_j - A_{ji}(x)\zeta_j.$$

In view of (H_1) and (H_2) the functions A_{ij}, B_{ij}, C_{ij} are of class C^1 on ab; moreover, the matrices $\| B_{ij} \|$ and $\| C_{ij} \|$ are symmetric and $\| B_{ij} \|$ is of rank $n - m$ on ab.

The system (2.4) is therefore equivalent to the system

$$(2.4') \qquad \begin{aligned} &L_i[\eta, \zeta] \equiv \eta'_i - A_{ij}(x)\eta_j - B_{ij}(x)\zeta_j = 0, \\ &L_{n+i}[\eta, \zeta|\lambda] \equiv \zeta'_i - C_{ij}(x)\eta_j + A_{ji}(x)\zeta_j \\ &\qquad - \int_a^b M_{ij}(x, t)\eta_j(t)\,dt + \lambda\eta_i = 0. \end{aligned}$$

Now if $c_i = c_{i\theta}$, $d_i = d_{i\theta}$ $(\theta = 1, \cdots, 2n - p)$ are linearly independent solutions of the equations

$$\Psi_{\gamma;ia} c_i + \Psi_{\gamma;ib} d_i = 0 \qquad (\gamma = 1, \cdots, p),$$

the boundary conditions (2.5) are equivalent to the linearly independent set

$$(2.5') \qquad \begin{aligned} s_\gamma[\eta, \zeta] &\equiv \Psi_\gamma[\eta(a), \eta(b)] = 0, \\ s_{p+\theta}[\eta, \zeta] &\equiv c_{i\theta}\{Q_{ia}[\eta] - \zeta_i(a)\} + d_{i\theta}\{Q_{ib}[\eta] + \zeta_i(b)\} = 0. \end{aligned}$$

For brevity, we shall speak of the boundary value problem (2.4), (2.5), or the equivalent problem (2.4'), (2.5'), as the boundary value problem B. Corresponding to these two forms of B, the term characteristic solution will be used to denote a set η_i, μ_a or a set η_i, ζ_i, where in each case the functions of the set are not all identically zero on ab, and the set satisfies for a corresponding value λ the integro-differential equations and two-point boundary conditions of B. If λ is a value for which there exist q linearly independent characteristic solutions, this value will be called a characteristic value of B of index q.

The above assumption (H_3) is equivalent to the assumption that there is no characteristic solution η_i, ζ_i of B for which $\eta_i \equiv 0$ on ab (see Bliss [3], p. 693; [4], p. 48). As is customary (see Reid [20], p. 575), we shall say that the order of anormality on the interval ab of the integro-differential equations of B is equal to r if on this interval there are exactly r linearly independent solutions $\eta_i \equiv 0$, $\zeta_i = v_{ih}(x)$ $(h = 1, \cdots, r)$ of (2.4'). It is to be noted that the value of r is independent of λ. It follows readily that $r \leq m$; moreover, if η is an arbitrary differentially admissible arc and x_1, x_2 are any two points on ab, then

$$(2.9) \qquad v_{ih}(x)\eta_i(x)\Big|_{x_1}^{x_2} = 0 \qquad (h = 1, \cdots, r).$$

As a consequence of these relations, we have that for a system of the above type which satisfies (H_1) and (H_2) the additional hypothesis (H_3) is equivalent to the assumption that the matrix

$$(2.10) \qquad \left\| \begin{matrix} \Psi_{\gamma;ja} & \Psi_{\gamma;jb} \\ -v_{jh}(a) & v_{jh}(b) \end{matrix} \right\| \qquad (j = 1, \cdots, n;\ h = 1, \cdots, r;\ \gamma = 1, \cdots, p)$$

has rank $p + r$. This implies, in particular, that $p \leq 2n - r$.

Now the class of admissible arcs η for a problem B is unchanged when the conditions $\Psi_\gamma = 0$ are replaced by conditions of the form

$$(2.11) \qquad \Psi_\gamma + e_{\gamma h}[-v_{jh}(a)\eta_j(a) + v_{jh}(b)\eta_j(b)] = 0 \qquad (\gamma = 1, \cdots, p),$$

where $e_{\gamma h}$ $(\gamma=1,\cdots,p;\ h=1,\cdots,r)$ are arbitrary constants. If η_i, ζ_i is a characteristic solution of the original problem B, there are constants e_h such that $\eta_i, \zeta_i + v_{ih}e_h$ is a characteristic solution of the modified problem B involving the end-conditions (2.11). Consequently, we shall not distinguish between two boundary value problems B_1 and B_2 which satisfy the above hypotheses and differ only in the end-conditions $\Psi_\gamma{}^1=0$ and $\Psi_\gamma{}^2=0$ $(\gamma=1,\cdots,p)$, respectively, and these conditions are such that the matrix

$$(2.12)\qquad \left\| \begin{matrix} \Psi^1{}_{\gamma;ja} & \Psi^1{}_{\gamma;jb} \\ \Psi^2{}_{\gamma;ja} & \Psi^2{}_{\gamma;jb} \\ -v_{jh}(a) & v_{jh}(b) \end{matrix} \right\| \quad (j=1,\cdots,n;\ \gamma=1,\cdots,p;\ h=1,\cdots,r)$$

has rank $p+r$.

If a problem B satisfies (H_1) and (H_2), but not (H_3), then the matrix (2.10) has rank $p+r-k$, where $0<k\leq p$. By deleting k of the end-conditions $\Psi_\gamma=0$ $(\gamma=1,\cdots,p)$ one may then obtain a problem B^1 which satisfies hypothesis (H_3), and which is equivalent to B in the sense that an arc η is admissible for B^1 if and only if it is admissible for B. Such a problem B^1 will be called the *normal boundary problem determined by the end-conditions* $\Psi_\gamma=0$ $(\gamma=1,\cdots,p)$.

For brevity, we shall refer to hypotheses (H_1), (H_2) and (H_3) as simply hypotheses (H).

3. Properties of the boundary value problem B. For brevity we shall set

$$M[u;v]=M[v;u]=\int_a^b\int_a^b u_i(x)M_{ij}(x,t)v_j(t)\,dxdt;$$
$$K[u;v]=K[v;u]=\int_a^b u_i(x)v_i(x)\,dx$$
$$Q[u;v]=Q[v;u]=(1/2)(u_i(a)Q_{ia}[v]+u_i(b)Q_{ib}[v]);$$
$$I[u;v]=I[v;u]$$
$$=Q[u;v]+\int_a^b[u'_i\omega_{\pi_i}(x,v,v')+u_i\omega_{\eta_i}(x,v,v')]dx+M[u;v];$$
$$M[u]=M[u;u];\quad K[u]=K[u;u];\quad Q[u]=Q[u;u];\quad I[u]=I[u;u];$$
$$I[u;v|\lambda]=I[u;v]-\lambda K[u;v],\quad I[u|\lambda]=I[u]-\lambda K[u].$$

In view of (2.6) and (2.7) we have:

LEMMA 3.1. *If η is a differentially admissible arc and μ_a an arbitrary set of functions of class D, the functions ζ_i defined by* (2.6) *are such that*

$$(3.1)\qquad \int_a^b 2\omega(x,\eta,\eta')\,dx=\int_a^b[B_{ij}(x)\zeta_i\zeta_j+C_{ij}(x)\eta_i\eta_j]\,dx.$$

The following properties of a boundary problem B satisfying hypotheses (H) will be stated without proof, since they follow as for the differential problem to which B reduces when $M_{ij}(x,t) \equiv 0$ (see, in particular, Hu [9] and Reid [18]).

THEOREM 3.1. *If η_i, ζ_i is a solution of the non-homogeneous system*

$$(3.2) \qquad L_i[\eta,\zeta] = 0, \quad L_{n+i}[\eta,\zeta|\lambda] + f_i(x) = 0, \quad s_\sigma[\eta,\zeta] = 0,$$

and u_i is an arbitrary differentially admissible arc, then $I[\eta; u|\lambda] - K[f; u] = 0$.

COROLLARY 1. *If η_i, ζ_i is a solution of (3.2), then $I[\eta|\lambda] - K[f;\eta] = 0$.*

COROLLARY 2. *If η_i, ζ_i is a characteristic solution of B corresponding to a characteristic value λ, then $I[\eta|\lambda] = 0$.*

THEOREM 3.2. *If η_i, ζ_i and η^*_i, ζ^*_i are solutions of system (3.2) for sets of functions $f_i(x), f^*_i(x)$, respectively, then $K[\eta; f^*] - K[\eta^*; f] = 0$.*

COROLLARY. *If η_i, ζ_i and η^*_i, ζ^*_i are characteristic solutions of B for distinct characteristic values λ and λ^*, then $K[\eta;\eta^*] = 0$.*

THEOREM 3.3. *The boundary problem B has only real characteristic values, and the corresponding characteristic solutions may be chosen real.*

LEMMA 3.2. *There exist positive constants m_0, l_0 such that for arbitrary differentially admissible arcs η we have*

$$(3.3) \qquad I[\eta] \geq \int_a^b [m_0\eta'_i\eta'_i - l_0\eta_i\eta_i]dx.$$

Let c be a constant such that $|2Q[\eta_{ia},\eta_{ib}]| \leq c[\eta_{ia}\eta_{ia} + \eta_{ib}\eta_{ib}]$, and choose a function $\phi(x)$ of class C^1 such that $\phi(b) = -1$, $\phi(a) = 1$. Moreover denote by l_1 a value such that $|M[\eta]| \leq l_1 K[\eta]$ for arbitrary continuous functions $\eta_i(x)$. It follows from Schwarz' inequality that l_1 may be chosen as $M_0(b-a)$, where M_0 is such that $|M_{ij}(x,t)| \leq M_0/n$ $(i,j = 1,\cdots,n;$ $a \leq x, t \leq b)$. Then for differentially admissible arcs η we have

$$(3.4) \qquad I[\eta] \geq \int_a^b 2\omega^*(x,\eta,\eta')dx,$$

when $2\omega^* = 2\omega + c[\phi'(x)\eta_i\eta_i + 2\phi(x)\eta_i\eta'_i] - l_1\eta_i\eta_i$. For a given differentially admissible η and arbitrary multipliers μ_α of class D, let A^*_{ij}, B^*_{ij}, C^*_{ij} denote the coefficients in (2.7) and (2.8) when ω is replaced by ω^* in equations (2.6). It is seen that $B^*_{ij} = B_{ij}$. It is then a consequence of Lemma 3.1 and (3.4) that

$$I[\eta] \geq \int_a^b [B_{ij}(x)\zeta_i\zeta_j + C^*_{ij}(x)\eta_i\eta_j]dx.$$

For arbitrary functions ζ_i the quantities $p_i = B_{ij}\zeta_j$ are such that $p_i\omega_{\pi_i\pi_j}p_j = B_{ij}\zeta_i\zeta_j$, and $\Phi_{\alpha\pi_j}p_j = 0$ $(\alpha = 1, \cdots, m)$. In view of (H_2) there is a positive constant l such that $B_{ij}\zeta_i\zeta_j \geq lp_ip_i$. Now $p_i = B_{ij}\zeta_j = \eta'_i - A^*_{ij}\eta_j$, and therefore $p_ip_i \geq (1/2)\eta'_i\eta'_i - A^*_{ki}A^*_{kj}\eta_i\eta_j$.[5] Inequality (3. 3) is then seen to be valid if $m_0 = l/2$, and l_0 is a constant such that for $a \leq x \leq b$ the relation $|[C^*_{ij} - lA^*_{ki}A^*_{kj}]u_iu_j| \leq l_0u_iu_i$ holds for arbitrary sets u_i.

Corollary. *There exists a constant λ_0 such that $I[\eta|\lambda_0] > 0$ for arbitrary non-identically vanishing differentially admissible arcs η.*

Lemma 3. 3. *If η_i, ζ_i is a characteristic solution of B for a value λ which is normed so that $K[\eta] = 1$, then*

(3. 5) $\quad \eta_i(x)\eta_i(x) \leq 2/(b-a) + 2(\lambda + l_0)(b-a)/m_0, \quad a \leq x \leq b,$

where m_0, l_0 are the constants of Lemma 3. 2.

If x_1, x are two points of ab, by Schwarz' inequality we have

$$\begin{aligned}[\eta_i(x) - \eta_i(x_1)]^2 = [\int_{x_1}^{x} \eta'_i(t)dt]^2 &\leq |x - x_1| \; |\int_{x_1}^{x} \eta'^2_i dt| \\ &\leq (b-a)\int_a^b \eta'^2_i dt, \quad (i = 1, \cdots, n),\end{aligned}$$

and hence, by Lemma 3. 2,

$$\begin{aligned}\sum_{i=1}^{n} [\eta_i(x) - \eta_i(x_1)]^2 \leq (b-a)\int_a^b \eta'_i\eta'_i dt &\leq [(b-a)/m_0](I[\eta] + l_0K[\eta]) \\ &\leq [(b-a)/m_0](\lambda + l_0).\end{aligned}$$

Since $K[\eta] = 1$ we may choose x_1 such that $\eta_i(x_1)\eta_i(x_1) \leq 1/(b-a)$. Relation (3. 5) is then a ready consequence of the elementary inequality

$$\eta_i(x)\eta_i(x) \leq 2\{\eta_i(x_1)\eta_i(x_1) + \sum_{i=1}^{n} [\eta_i(x) - \eta_i(x_1)]^2\}.$$

The remainder of the present section depends upon the results of Section 8. According to the definition of that section, the integro-differential system B may be shown to be self-adjoint with respect to the constant transformation matrix

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[5] This relation is an immediate consequence of the elementary inequality

$$a^2 \leq 2[(a-b)^2 + b^2],$$

which is satisfied by arbitrary real constants a and b.

$$(3.6)\qquad \| T_{\sigma\tau} \| = \begin{Vmatrix} 0_{ij} & \delta_{ij} \\ -\delta_{ij} & 0_{ij} \end{Vmatrix} \qquad (i, j = 1, \cdots, n;\ \sigma, \tau = 1, \cdots, 2n).$$

If λ is not a characteristic value of B, it follows that the non-homogeneous system

$$(3.7)\qquad L_i[\eta, \zeta] = 0,\ L_{n+i}[\eta, \zeta|\lambda] = k_i(x), \qquad s_\sigma[\eta, \zeta] = 0$$

has for given continuous functions $k_i(x)$ a unique solution η_i, ζ_i. Moreover, this solution is given by

$$(3.8)\qquad \eta_i(x|\lambda) = \int_a^b G^2{}_{ij}(x, t|\lambda) k_j(t)\, dt,\ \zeta_i(x|\lambda) = \int_a^b G^4{}_{ij}(x, t|\lambda) k_j(t)\, dt,$$

where

$$(3.9)\qquad \| G_{\sigma\tau}(x, t|\lambda) \| = \begin{Vmatrix} G^1{}_{ij}(x, t|\lambda) & G^2{}_{ij}(x, t|\lambda) \\ G^3{}_{ij}(x, t|\lambda) & G^4{}_{ij}(x, t|\lambda) \end{Vmatrix}$$

is the Green's matrix of the system (2.4′), (2.5′) for this value of λ.

4. Existence and properties of characteristic values. There will now be defined a sequence of classes of arcs, and we shall consider the problem of minimizing $I[\eta]$ in each of these classes. The class S_1 is defined as the totality of admissible arcs which satisfy the relation $K[\eta] = 1$. In view of (H_3) the class S_1 is non-vacuous (see Bliss [3], p. 694). We shall first prove the following result

THEOREM 4.1. *Suppose the boundary problem B satisfies hypotheses (H), and denote by Λ_1 the greatest lower bound of $I[\eta]$ in the class S_1. Then $\lambda = \Lambda_1$ is the smallest characteristic value of B.*

In view of Lemma 3.2 the greatest lower bound of $I[\eta]$ in S_1, Λ_1, is finite. It also follows from Corollary 2 to Theorem 3.1 that there is no characteristic value of B less than Λ_1. It will now be proved by indirect argument that $\lambda = \Lambda_1$ is a characteristic value. Let $u_\nu = [u_{i\nu}]$ $(\nu = 1, 2, \cdots)$ be a sequence of arcs belonging to S_1 and such that $I[u_\nu] \to \Lambda_1$ as $\nu \to \infty$.

Now consider the non-homogeneous system

$$(4.1)\qquad L_i[\eta, \zeta] = 0, \qquad L_{n+i}[\eta, \zeta|\Lambda_1] = u_{i\nu}(x), \qquad s_\sigma[\eta, \zeta] = 0,$$

that is, the system (3.7) for $\lambda = \Lambda_1$, $k_i = u_{i\nu}$. If $\lambda = \Lambda_1$ is not a characteristic value of B the system (4.1) has a unique solution $\eta_i = U_{i\nu}(x)$, $\zeta_i = V_{i\nu}(x)$, $(\nu = 1, 2, \cdots)$, given by the corresponding relations (3.8). It is a consequence of elementary inequalities that there exist constants l_1 and l_2 depending solely upon the value of Λ_1 and such that

$$(4.2) \qquad |U_{i\nu}(x)| \leq l_1 K[u_\nu] + l_2 = l_1 + l_2 \qquad (\nu = 1, 2, \cdots).$$

Now set

$$(4.3) \qquad w_{i\nu}(x) = u_{i\nu}(x) + cU_{i\nu}(x) \qquad (i = 1, \cdots, n;\ \nu = 1, 2, \cdots),$$

where c is an arbitrary real constant. It is readily verified with the aid of Theorem 3. 1 that

$$(4.4) \qquad I[w_\nu|\Lambda_1] = I[u_\nu|\Lambda_1] - 2c - c^2 K[u_\nu; U_\nu].$$

By Schwarz' inequality we have $(K[u_\nu; U_\nu])^2 \leq K[u_\nu]K[U_\nu] = K[U_\nu]$, and it therefore follows by (4. 2) that there is a constant $l_3 > 0$ such that $|K[u_\nu; U_\nu]| \leq l_3$ $(\nu = 1, 2, \cdots)$. Now $I[u_\nu|\Lambda_1] = I[u_\nu] - \Lambda_1$ tends to zero as $\nu \to \infty$. Hence if c be chosen so that $0 < c < 2/l_3$, it follows from (4. 4) that $I[w_\nu|\Lambda_1] = I[w_\nu] - \Lambda_1 K[w_\nu]$ is negative for ν sufficiently large. This, however, contradicts the assumption that Λ_1 is the greatest lower bound of $I[\eta]$ in S_1. Hence $\lambda = \Lambda_1$ is a characteristic value of B, and Theorem 4. 1 is proved.

Now suppose classes $S_1, \cdots, S_{s-1}$ $(s \geq 2)$ have been defined, each of these classes is non-vacuous, and that $\lambda = \Lambda_t$ $(t = 1, \cdots, s-1)$, where Λ_t is the greatest lower bound of $I[\eta]$ in the class S_t, is a characteristic value of B of index ρ_t. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\rho$ $(\rho = \rho_1 + \rho_2 + \cdots + \rho_{s-1})$ denote these characteristic values each repeated a number of times equal to its index, and choose $\eta_i = y_{i\kappa}(x)$, $\zeta_i = z_{i\kappa}(x)$ as a characteristic solution of B for $\lambda = \lambda_\kappa$ $(\kappa = 1, \cdots, \rho)$. These solutions will be supposed to be orthonormal in the sense that $K[y_\kappa; y_{\kappa'}] = \delta_{\kappa\kappa'}$ $(\kappa, \kappa' = 1, \cdots, \rho)$. The class S_s is then defined as the sub-class of S_{s-1} consisting of all admissible arcs η which satisfy the relations

$$(4.5) \qquad K[y_\kappa; \eta] = 0 \qquad (\kappa = 1, \cdots, \rho)$$

The class S_s is well defined and non-vacuous (see Reid [19], p. 847). We shall now prove the following induction theorem.

THEOREM 4. 2. *Suppose the boundary problem B satisfies hypotheses (H), and denote by Λ_s the greatest lower bound of $I[\eta]$ in the class S_s. Then $\lambda = \Lambda_s$ is a characteristic value of B, and $\Lambda_s > \Lambda_{s-1}$.*

To prove this theorem we shall consider the auxiliary problem of minimizing the expression $\boldsymbol{I}[\boldsymbol{\eta}] = I[\eta|\Lambda_s]$ in the class $\boldsymbol{S}_1$ of arcs

$$\boldsymbol{\eta} = (\eta_1, \cdots, \eta_n, \eta_{n+1}, \cdots, {}_{n+\rho}),$$

where the functions $\eta_1, \cdots, \eta_{n+\rho}$ are of class D^1 on ab, satisfy the differential equations

$$(1.3^*)\qquad \Phi_{\alpha\pi_i}(x)\eta'_i + \Phi_{\alpha\eta_i}(x)\eta_i = 0, \qquad \eta'_{n+\kappa} - y_{i\kappa}(x)\eta_i = 0$$
$$(\alpha = 1,\cdots, m;\ \kappa = 1,\cdots,\rho;\ i = 1,\cdots, n),$$

the end conditions

$$(1.4^*)\qquad \Psi_{\gamma;ia}\eta_i(a) + \Psi_{\gamma;ib}\eta_i(b) = 0, \qquad \eta_{n+\kappa}(a) - \eta_{n+\kappa}(b) = 0$$
$$(\gamma = 1,\cdots, p;\ \kappa = 1,\cdots,\rho),$$

and the norming condition

$$(1.5)\qquad \int_a^b [\eta_i\eta_i + \eta_{n+\kappa}\eta_{n+\kappa}]dx \equiv \boldsymbol{K}[\boldsymbol{\eta}] = 1.$$

If $\boldsymbol{B}$ denote the corresponding boundary value problem involving

$$\boldsymbol{\eta} = (\eta_1,\cdots,\eta_{n+\rho}), \qquad \boldsymbol{\zeta} = (\zeta_1,\cdots,\zeta_{n+\rho})$$

it is readily seen that $\boldsymbol{B}$ satisfies the corresponding hypotheses (H). Consequently, if $\boldsymbol{\Lambda}_1$ is the greatest lower bound of $\boldsymbol{I}[\boldsymbol{\eta}]$ in the class $\boldsymbol{S}_1$, it follows from the above theorem that $\boldsymbol{\Lambda}_1$ is the smallest characteristic value of $\boldsymbol{B}$. Suppose η is an arbitrary admissible arc for B which satisfies relations (4.5), and let $\boldsymbol{\eta}$ denote the corresponding admissible arc for $\boldsymbol{B}$. In view of the relation $I[\eta|\Lambda_s] = \boldsymbol{I}[\boldsymbol{\eta}] \geq \boldsymbol{\Lambda}_1\boldsymbol{K}[\boldsymbol{\eta}] \geq \boldsymbol{\Lambda}_1 K[\eta]$, and the definition of Λ_s, it follows that $\boldsymbol{\Lambda}_1 = 0$. Hence there exist functions $\eta_1,\cdots,\eta_{n+\rho}, \zeta_1,\cdots,\zeta_{n+\rho}$ not all zero and such that

$$(4.6)\qquad \begin{aligned} &L_i[\eta,\zeta] = 0, \quad \eta'_{n+\kappa} = y_{i\kappa}(x)\eta_i, \quad \zeta'_{n+\kappa} = 0,\\ &L_{n+i}[\eta,\zeta|\Lambda_s] - y_{i\kappa}(x)\zeta_{n+\kappa} = 0,\\ &s_\sigma[\eta,\zeta] = 0, \quad \eta_{n+\kappa}(a) - \eta_{n+\kappa}(b) = 0,\\ &\qquad (i = 1,\cdots,n;\ \sigma = 1,\cdots,2n;\ \kappa = 1,\cdots,\rho).\end{aligned}$$

It follows from the differential equations and boundary conditions satisfied by the functions $y_{i\kappa}$, $z_{i\kappa}$, together with Theorem 3.2, that the constants $\zeta_{n+\kappa}$ are all zero, and consequently the functions η_i, ζ_i are not all identically zero on ab. Hence $\lambda = \Lambda_s$ is a characteristic value of B corresponding to which there are characteristic solutions η_i, ζ_i which satisfy relations (4.5). This last condition implies $\Lambda_s \neq \Lambda_{s-1}$, and therefore $\Lambda_s > \Lambda_{s-1}$. This completes the proof of Theorem 4.2.

For a given boundary value problem satisfying hypotheses (H) we shall denote by $\{\lambda_s\}$ $(s = 1, 2,\cdots)$ the totality of characteristic values, each repeated a number of times equal to its index, and the entire set ordered so that

$$(4.7)\qquad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s \leq \cdots.$$

It will also be supposed that $\eta_i = y_{is}(x)$, $\zeta_i = z_{is}(x)$ is a solution of B for

$\lambda = \lambda_s$ $(s = 1, 2, \cdots)$, and that these solutions are orthonormal in the sense that

$$K[y_s; y_t] = \delta_{st} \qquad (s, t = 1, 2, \cdots). \tag{4.8}$$

Finally, if we denote by $\mathfrak{S}_s$ the totality of admissible arcs η which satisfy the relations $K[\eta] = 1$, $K[y_\kappa; \eta] = 0$ $(\kappa = 1, \cdots, s-1)$, it is a consequence of the above theorems that λ_s is the minimum of $I[\eta]$ in the class $\mathfrak{S}_s$.

The following theorem is essentially the so-called maximum-minimum property of the characteristic values of B (see Courant-Hilbert [7], Chapter VI).

THEOREM 4.3. *If s is a given positive integer and d_κ $(\kappa = 1, \cdots, s)$ are arbitrary real constants such that $d_1{}^2 + \cdots + d_s{}^2 = 1$, then the admissible arc $\eta_i = y_{i1}(x)d_1 + \cdots + y_{is}(x)d_s$ satisfies the relation $I[\eta] \leq \lambda_s$.*

For such an arc η it follows with the use of Theorem 3.1 that

$$I[\eta] = \sum_{\kappa=1}^{s} \lambda_\kappa d_\kappa{}^2 \leq \lambda_s \sum_{\kappa=1}^{s} d_\kappa{}^2 = \lambda_s. \tag{4.9}$$

THEOREM 4.4. *Suppose u is an admissible arc such that $K[u] = 1$, and that there is a positive integer s such that $I[u] = \lambda_s$, $K[y_\kappa; u] = 0$ $(\kappa = 1, \cdots, s-1)$. Then the functions $u_i(x)$ are of class C^1 and there exist functions $v_i(x)$ of class C^1 such that $\eta_i = u_i$, $\zeta_i = v_i$ is a characteristic solution of B for $\lambda = \lambda_s$.*

Let N be the greatest positive integer such that $\lambda_N = \lambda_s$. For

$$\eta_i = u_i - \sum_{\kappa=1}^{N} y_{i\kappa}(x) K[y_\kappa; u] = u_i - \sum_{\kappa=s}^{N} y_{i\kappa}(x) K[y_\kappa; u]$$

we have $K[y_\kappa; u] = 0$ $(\kappa = 1, \cdots, N)$. Now suppose that the functions η_i are not all identically zero on ab. Then

$$0 < K[\eta] = 1 - \sum_{\kappa=s}^{N} (K[y_\kappa; u])^2.$$

It is then a consequence of the minimizing property of λ_{N+1} that

$$I[\eta] \geq \lambda_{N+1} K[\eta] = \lambda_{N+1}\{1 - \sum_{\kappa=s}^{N} (K[y_\kappa; u])^2\}. \tag{4.10}$$

On the other hand, since $y_{i\kappa}$, $z_{i\kappa}$ $(\kappa = s, \cdots, N)$ is a solution of B for $\lambda = \lambda_\kappa = \lambda_s$, we have by the use of Theorem 3.1 that

$$I[\eta] = I[u] - \lambda_s \sum_{\kappa=s}^{N} (K[y_\kappa; u])^2 = \lambda_s\{1 - \sum_{\kappa=s}^{N} (K[y_\kappa; u])^2\}. \tag{4.11}$$

Relations (4.10) and (4.11) are contradictory since $\lambda_{N+1} > \lambda_s$. Hence $\eta_i(x) \equiv 0$, and the set u_i, $v_i = \sum_{\kappa=s}^{N} z_{i\kappa}(x) K[y_\kappa; u]$ satisfies the conclusion of the above theorem.

In closing this section, we shall indicate by simple examples that certain important properties of differential systems are not in general satisfied by the integro-differential system (2.4′), (2.5′). Firstly, it is in general not true that for a given λ there exists a solution η_i, ζ_i of the equations (2.4′) assuming prescribed initial values at $x = a$. This fact is illustrated by the example: $n = 1$, $Q \equiv 0$, $2\omega \equiv \eta'^2$, $M(x, t) = x + t$, $\lambda = 0$, $a = 0$, and b the positive root of the equation $2880 - 480b^4 - b^8 = 0$. We have $\eta' = \zeta$, and (2.4′) is equivalent to $\eta'' - \int_0^b (x + t)\eta(t)\,dt = 0$. It follows readily that any solution of this equation must be of the form $\alpha + \beta x + \gamma x^2 + \delta x^3$, and upon substitution it is found that for b a root of the above equation there exists a solution η if and only if α and β satisfy a linear relation. Moreover, $\eta = 4b^5 x^2 + (40 - 5b^4)x^3$ is a solution such that $\eta(0) = 0 = \eta'(0)$.

Secondly, consider the example $n = 1$, $Q \equiv 0$, $2\omega \equiv \eta'^2 - \eta^2$, $M(x, t) = 1/4\pi$, $\lambda = 0$, $a = 0$, $b = 4\pi$, $\Psi_1 \equiv \eta(a)$, $\Psi_2 \equiv \eta(b)$. The system (2.4′), (2.5′) is then equivalent to $\eta'' + \eta - (1/4\pi)\int_0^{4\pi} \eta(t)\,dt = 0$, $\eta(0) = 0 = \eta(4\pi)$. The integro-differential equation has the non-vanishing solution $\eta \equiv 1$, yet the corresponding functional $I[\eta]$ is negative for certain η's satisfying the end-conditions, for example, $I[\sin x/2] = -3\pi/2$.

It is to be noted, however, that in case $M_{ij}(x, t)$ is of product form $M_{ij}(x, t) \equiv \phi_{i1}(x)\phi_{j1}(t) + \cdots + \phi_{ig}(x)\phi_{jg}(t)$ the general integro-differential system (2.4′), (2.5′) may be reduced to an equivalent differential problem by the introduction of g additional variables. Let

$$\eta_{n+s}(x) = \int_a^x \phi_{is}(t)\eta_i(t)\,dt, \qquad (s = 1, \cdots, g).$$

Then the functional (1.1) may be written as

$$(4.12) \qquad 2Q[\eta(a), \eta(b)] + \sum_{s=1}^{g} [\eta_{n+s}(b)]^2 + \int_a^b 2\omega(x, \eta, \eta')\,dx,$$

where the symbol η still denotes the original set $\eta_1, \cdots, \eta_n$. Now consider the resulting differential problem consisting of the Euler equations and transversality conditions associated with (4.12) and the conditions (1.3), (1.4), (1.5) on the functions $\eta_1, \cdots, \eta_n$, together with the additional conditions $\Phi_{m+s} \equiv \eta'_{n+s} - \phi_{is}(x)\eta_i(x) = 0$, $\Psi_{p+s} \equiv \eta_{n+s}(a) = 0$, $(s = 1, \cdots, g)$. This

system is equivalent to the original problem. It is to be noted however, that for this new problem the norming form (1. 5) does not involve all of the variables $\eta_1, \cdots, \eta_{n+g}$ and hence the associated boundary problem is not of the form (2. 4′), (2. 5′); it is, however, of the form (7. 7) considered in Section 7. The example of the preceding paragraph is of this type, and consequently we see why in general for the case of a single integro-differential equation of the second order with the boundary conditions $\eta(a) = 0 = \eta(b)$ the existence of a non-vanishing solution could not be expected to imply the non-negativeness of the associated functional $I[\eta]$.

5. Comparison theorems. A boundary problem B of the type treated in this paper depends upon differential forms $\omega(x,\eta,\eta'), \Phi_\alpha(x,\eta,\eta')$ $(\alpha = 1, \cdots, m)$, an end-form $Q[\eta(a), \eta(b)]$, end-relations $\Psi_\gamma[\eta(a), \eta(b)]$ $(\gamma = 1, \cdots, p)$, and functions $M_{ij}(x, t)$. In this section we shall prove some comparison theorems for such a problem B and a second problem B^* involving corresponding quantities $\omega^*(x, \eta, \eta'), \Phi_\alpha(x, \eta, \eta')$ $(\alpha = 1, \cdots, m)$, $Q^*[\eta(a), \eta(b)]$, $\Psi^*_\nu[\eta(a), \eta(b)]$ $(\nu = 1, \cdots, p^*)$, $M^*_{ij}(x, t)$. The expressions (1. 1) for B and B^* will be denoted by $I[\eta]$ and $I^*[\eta]$, respectively. We shall assume that B and B^* involve the same auxiliary differential equations $\Phi_\alpha = 0$ $(\alpha = 1, \cdots, m)$, and that each of these problems satisfies hypotheses (H) of Section 2. The set of characteristic values and corresponding characteristic solutions of B and B^* will be denoted by $\lambda_s, \eta_i = y_{is}(x)$, $\zeta_i = z_{is}(x)$ and λ^*_s, $\eta_i = y^*_{is}(x)$, $\zeta_i = z^*_{is}(x)$ $(s = 1, 2, \cdots)$, respectively. Each of the sets of characteristic values is supposed to be ordered as in (4. 7), and for each of the problems the characteristic solutions are chosen orthonormal in the sense of (4. 8). The classes of admissible arcs $\mathfrak{S}_s, \mathfrak{S}^*_s$ $(s = 1, 2, \cdots)$ are defined for the two problems as described in the preceding section.

As indicated in the introduction, when $M_{ij}(x, t) \equiv 0$, the theorems of this section are essentially those previously given by Morse [16] and Hu [9]. So far as the author knows, however, for even the simplest type of integro-differential system of the form here considered when the functions M_{ij} are not all identically zero, these theorems are new. As indicated by the examples at the end of Section 4, integro-differential systems of the type here considered do not in general possess some of the well known properties of differential systems. Hence the fact that such integro-differential systems do possess the same type of comparison and oscillation theorems as corresponding differential boundary problems is of significance. In Section 7 the results of the present section will be used to prove corresponding theorems for integro-differential systems involving the characteristic parameter in a more general fashion. It

is to be emphasized that in the proof of the results of this and the following section, *no assumption of normality on sub-intervals is made.*

We shall first prove the following general comparison theorem.

THEOREM 5.1. *Suppose that B and B^* have in common $\Phi_\alpha(x, \eta, \eta')$ $(\alpha = 1, \cdots, m)$, $\Psi_\gamma[\eta(a), \eta(b)]$ $(\gamma = 1, \cdots, p)$, and that $I^*[\eta] \geqq I[\eta]$ for arbitrary admissible arcs η. Then $\lambda^*_s \geqq \lambda_s$ $(s = 1, \cdots)$.*

Corresponding to a positive integer s, let $d_1, \cdots, d_s$ be real constants such that $d_1^2 + \cdots d_s^2 = 1$, and the arc $y_i = y^*_{i1}d_1 + \cdots + y^*_{is}d_s$ satisfies the relations $K[y_\kappa; \eta] = 0$ $(\kappa = 1, \cdots, s-1)$. The orthonormal relations satisfied by the characteristic solutions of B^* imply $K[\eta] = 1$, and hence η belongs to $\mathfrak{S}_s$. The inequalities

$$\lambda^*_s \geqq I^*[\eta] \geqq I[\eta] \geqq \lambda_s$$

are then consequences of Theorem 4.3 and the minimizing property of λ_s.

The following corollary is immediate

COROLLARY 1. *If the hypotheses of Theorem 5.1 are strengthened so that $I^*[\eta] > I[\eta]$ for arbitrary non-identically vanishing admissible arcs η, then $\lambda^*_s > \lambda_s$ $(s = 1, 2, \cdots)$.*

COROLLARY 2. *For a given problem B, let B^0 denote a particular related problem involving the same end-form Q and end conditions $\Psi_\gamma = 0$ $(\gamma = 1, \cdots, p)$, for which $M^0_{ij}(x, t) \equiv 0$, and $2\omega^0[x, \eta, \eta'] \equiv 2\omega(x, \eta, \eta') - l_0\eta_i\eta_i$, where l_0 is a constant such that $M[\eta] \leqq l_0K[\eta]$ for arbitrary continuous arcs $\eta \equiv [\eta_i(x)]$. If $\{\lambda_s^0\}$ denote the ordered set of characteristic values of B^0, then $\lambda_s \geqq \lambda_s^0$ $(s = 1, 2, \cdots)$.*

Suppose two problems B and B^* have in common $\Phi_\alpha(x, \eta, \eta')$ $(\alpha = 1, \cdots, m)$ and $\Psi_\gamma[\eta(a), \eta(b)]$ $(\gamma = 1, \cdots, p)$. We shall denote by the difference problem [6] $D(B, B^*)$ the problem involving these same relations, and the expression $\Delta I[\eta] = I^*[\eta] - I[\eta]$. Hypotheses (H_1) and (H_3) for $D(B, B^*)$ are consequences of these hypotheses for B and B^*.

THEOREM 5.2. *Suppose hypotheses (H) are satisfied by each of the problems B, B^*, $D(B, B^*)$. Let $\lambda = \boldsymbol{\lambda}_s$, $\eta_i = \mathbf{y}_{is}(x)$, $\zeta_i = \mathbf{z}_{is}(x)$ $(s = 1, 2, \cdots)$ denote the characteristic values and characteristic solutions of $D(B, B^*)$, supposedly ordered as in (4.7) and orthonormal in the sense of (4.8). If s and t are arbitrary positive integers, then $\lambda^*_{s+t-1} \geqq \lambda_s + \boldsymbol{\lambda}_t$.*

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[6] This terminology is due to Morse [16], p. 100.

Let $d_1, \cdots, d_{s+t-1}$ be real constants such that the arc

$$\eta_i = y^*_{i1}d_1 + \cdots + y^*_{is+t-1}d_{s+t-1}$$

satisfies the $s + t - 2$ conditions

$$K[y_\kappa; \eta] = 0 \quad (\kappa = 1, \cdots, s-1), \qquad K[\mathbf{y}_\nu; \eta] = 0 \quad (\nu = 1, \cdots, t-1),$$

and the relation $K[\eta] = 1$. Then by Theorem 4.3 and the minimizing properties of λ_s and $\boldsymbol{\lambda}_t$ we have

$$\lambda^*_{s+t-1} \geq I^*[\eta] = I[\eta] + \Delta I[\eta] \geq \lambda_s + \boldsymbol{\lambda}_t.$$

COROLLARY. *Under the hypotheses of Theorem 5.2, if h_s denote the number of characteristic values of $D(B, B^*)$ less than λ_s, then $\lambda^*_{s+h_s} \geq 2\lambda_s$.*

We shall now consider two problems B and B^* which differ only in the end-forms Q and Q^*. Let

$$\begin{aligned} 2Q^* - 2Q &\equiv 2d[\eta(a), \eta(b)] \\ &= d_{ia;ja}\eta_i(a)\eta_j(a) + 2d_{ia;jb}\eta_i(a)\eta_j(b) + d_{ib;jb}\eta_i(b)\eta_j(b). \end{aligned}$$

The problems B and B^* involve the same system of integro-differential equations. Let r denote the order of anormality on ab of this system, as defined in Section 2, and suppose $\eta_i \equiv 0$, $\zeta_i = v_{ih}(x)$ $(h = 1, \cdots, r)$ are r linearly independent solutions of this system on ab. The comparison theorem to be established involves the number of positive and negative zeros of the determinant

$$(5.1) \qquad D(\rho) = \begin{vmatrix} d_{ia;jb} - \rho\delta_{ij} & d_{ia;jb} & \Psi_{\nu;ia} & -v_{ik}(a) \\ d_{ja;ib} & d_{ib;jb} - \rho\delta_{ij} & \Psi_{\nu;ib} & v_{ik}(b) \\ \Psi_{\gamma;ja} & \Psi_{\gamma;jb} & O_{\gamma\nu} & O_{\gamma k} \\ -v_{jh}(a) & v_{jh}(b) & O_{h\nu} & O_{hk} \end{vmatrix},$$

where $i, j = 1, \cdots, n$; $\gamma, \nu = 1, \cdots, p$; $h, k = 1, \cdots, r$.

The determinant $D(\rho)$ is a polynomial in ρ of degree $2n - p - r$. It is well known (see, for example, Caratheodory [IV], pp. 164-189) that the zeros of $D(\rho)$ are all real, and that if a given value is a zero of $D(\rho)$ of multiplicity q there are exactly q linearly independent solutions of the linear homogeneous equations in $2n + p + r$ unknowns whose coefficients are the elements of the rows of the matrix of $D(\rho)$. Let $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_{2n-p-r}$ denote these zeros of $D(\rho)$, and let $w = w_{lg}$ $(l = 1, \cdots, 2n + p + r;\ g = 1, \cdots, 2n - p - r)$ be a solution of this system of algebraic equations corresponding to $\rho = \rho_g$. The solutions may be chosen orthonormal in the sense that $w_{\sigma g}w_{\sigma g'} = \delta_{gg'}$

$(g, g' = 1, \cdots, 2n - p - r;\ \sigma = 1, \cdots, 2n)$. Moreover, if R is a given value such that $0 \leq R < 2n - p - r$, and η_{ia}, η_{ib} is a set of values such that

$$\begin{aligned} &\Psi_\gamma[\eta_a;\eta_b] = 0, \quad -v_{ih}(a)\eta_{ia} + v_{ih}(b)\eta_{ib} = 0 \\ (5.2) \qquad & \qquad\qquad\qquad\qquad\qquad\qquad (\gamma = 1, \cdots, p;\ h = 1, \cdots, r), \\ &w_{ig}\eta_{ia} + w_{n+i\,g}\eta_{ib} = 0 \qquad\qquad (g = 1, \cdots, R), \end{aligned}$$

then $2d[\eta_a;\eta_b] \geq \rho_{R+1}[\eta_{ia}\eta_{ia} + \eta_{ib}\eta_{ib}]$.

We now prove the following comparison theorem:

THEOREM 5.3. *Suppose the boundary problems B and B^* differ only in the end-forms Q and Q^*. Let N and P denote, respectively, the number of negative and positive zeros of $D(\rho)$ defined by (5.1) in terms of the difference end-form $d = Q^* - Q$, each zero being counted a number of times equal to its multiplicity. Then for an arbitrary positive integer s, $\lambda^*_{s+N} \geq \lambda_s$ and $\lambda_{s+P} \geq \lambda^*_s$.*

There clearly exist constants $d_1, \cdots, d_{s+N}$ such that $d_1^2 + \cdots + d^2_{s+N} = 1$, the arc $\eta_i = y^*_{i1}d_1 + \cdots + y^*_{is+N}d_{s+N}$ satisfies the relations $K[y_\kappa;\eta] = 0$ $(\kappa = 1, \cdots, s-1)$, and the set of end-values $\eta_i(a)$, $\eta_i(b)$ is orthogonal to each of the sets $[w_{\sigma g}]$ $(g = 1, \cdots, N)$ in the sense of (5.2). Since the arc η is admissible, its end-values also satisfy the other relations of (5.2), and consequently

$$(5.3) \qquad 2d[\eta(a), \eta(b)]) \geq \rho_{N+1}[\eta_i(a)\eta_i(a) + \eta_i(b)\eta_i(b)] \geq 0.$$

Finally, since $K[\eta] = 1$, the arc η belongs to $\mathfrak{S}_s$ and by Theorem 4.3 and the minimum property of λ_s we have

$$\lambda^*_{s+N} \geq I^*[\eta] = I[\eta] + 2d[\eta(a), \eta(b)] \geq I[\eta] \geq \lambda_s.$$

The remainder of the conclusion of the theorem is a ready consequence of the relation $I[\eta] = I^*[\eta] - 2d[\eta(a), \eta(b)]$, together with the fact that when the difference form d is replaced by its negative the zeros of $D(\rho)$ are also replaced by their negatives.

If Δ is a finite interval, either closed or open, of the λ-axis, we shall denote by $V[\Delta]$ the number of characteristic values of B on Δ. Corresponding to a value L we shall also denote by $V_L(W_L)$ the number of characteristic values of B which are less (not greater) than L. The numbers $V^*[\Delta]$, V^*_L, W^*_L are defined for B^* in an analogous manner.

The following corollary is an immediate consequence of the above theorem.

COROLLARY. *Under the hypotheses of Theorem 5.3, for every L we have*

$V_L - P \leq V^*_L \leq V_L + N$, $W_L - P \leq W^*_L \leq W_L + N$; moreover, $|V[\Delta] - V^*[\Delta]| \leq N + P$ for every finite sub-interval Δ of the λ-axis.

Now suppose that two problems B and B^* satisfy hypotheses (H), and differ only in the end-conditions $\Psi_\gamma[\eta(a), \eta(b)] = 0$ $(\gamma = 1, \cdots, p)$ and $\Psi^*_\nu[\eta(a), \eta(b)] = 0$ $(\nu = 1, \cdots, p^*)$. Corresponding to the terminology of Morse ([16], p. 92), we shall say that B^* is a sub-problem of B if the matrix

$$(5.4) \qquad \begin{Vmatrix} \Psi_{\gamma;\, ia} & \Psi_{\gamma;\, jb} \\ \Psi^*_{\nu;\, ja} & \Psi^*_{\nu;\, jb} \\ -v_{jh}(a) & v_{jh}(b) \end{Vmatrix} \quad (\gamma = 1, \cdots, p;\ \nu = 1, \cdots, p^*;\ h = 1, \cdots, r)$$

has rank $r + p^*$. If B^* is a sub-problem of B, clearly $p^* \geq p$. The number $p^* - p$ will be called the dimension of B^* as a sub-problem of B. It follows readily that if B^* is a sub-problem of B and η is an admissible arc for B^*, then η is also an admissible arc for B. Moreover, since (5.4) has rank $r + p^*$, if B^* is a sub-problem of B then in the class of differentially admissible arcs η the end-conditions $\Psi^*_\nu = 0$ $(\nu = 1, \cdots, p^*)$ are equivalent to the p conditions $\Psi_\gamma = 0$ $(\gamma = 1, \cdots, p)$ together with $p^* - p$ other end-conditions.

In view of the above remarks the following results are consequences of the minimizing properties of the characteristic values of B and B^*, together with the result of Theorem 4.3. Since the details of proof are the same as those used by Hu ([19], Section 15) to prove the corresponding results for the differential system he considered, these results will be stated without proof.

THEOREM 5.4. *If B^* is a sub-problem of B of dimension $p^* - p$, then* $\lambda_{s+p^*-p} \geq \lambda^*_s \geq \lambda_s$ $(s = 1, 2, \cdots)$.

COROLLARY. *Under the hypotheses of Theorem 5.4, for every L we have $V_L - (p^* - p) \leq V^*_L \leq V_L$, $W_L - (p^* - p) \leq W^*_L \leq W_L$; moreover, $|V[\Delta] - V^*[\Delta]| \leq p^* - p$ for every finite sub-interval Δ of the λ-axis.*

Let us now consider two problems B and B^* which have in common $\omega(x, \eta, \eta')$, $\Phi_\alpha(x, \eta, \eta')$, $M_{ij}(x, t)$ but involving, respectively, the sets

$$Q[\eta(a), \eta(b)], \qquad \Psi_\gamma[\eta(a), \eta(b)] \qquad (\gamma = 1, \cdots, p)$$

and

$$Q^*[\eta(a), \eta(b)], \qquad \Psi^*_\nu[\eta(a), \eta(b)] \qquad (\nu = 1, \cdots, p^*).$$

We suppose, as before, that each of the problems B and B^* satisfies hypotheses (H). Let $q + r$ denote the rank of the matrix (5.4). Clearly, $q \geq p$,

$q \geq p^*$, $q \leq p + p^*$. There are seen to exist q end-relations $\Psi_\mu{}^0[\eta(a), \eta(b)]$ $(\mu = 1, \cdots, q)$ satisfying (H_3) and such that the problem B_0 involving ω, Φ_a, M_{ij}, Q, $\Psi_\mu{}^0$ is a sub-problem of B, and the problem $B_0{}^*$ involving ω, Φ_a, M_{ij}, Q^*, $\Psi_\mu{}^0$ is a sub-problem of B^*. For brevity we shall say that the set of end-conditions $\Psi_\mu{}^0 = 0$ $(\mu = 1, \cdots, q)$ is the intersection of the sets $\Psi_\gamma = 0$ $(\gamma = 1, \cdots, p)$ and $\Psi^*_\nu = 0$ $(\nu = 1, \cdots, p^*)$ relative to the differential equations $\Phi_a = 0$. In comparing the two problems B_0 and B^*_0 use is made of the determinant (5.1) for these two problems. This determinant will be denoted by $D_0(\rho)$, and is seen to be a polynomial of degree $2n - r - q$ in ρ.

The following comparison theorem is the analogue of a theorem proved by Morse ([16], p. 94) for the differential system he considered, and is a consequence of the preceding theorems of this section.

THEOREM 5.5. *Suppose B and B^* have in common ω, Φ_a $(\alpha = 1, \cdots, m)$, $M_{ij}(x, t)$ and involve, respectively, the sets Q, Ψ_γ $(\gamma = 1, \cdots, p)$ and Q^*, Ψ^*_ν $(\nu = 1, \cdots, p^*)$. Suppose, moreover, that $\Psi_\mu{}^0 = 0$ $(\mu = 1, \cdots, q)$ is the intersection of $\Psi_\gamma = 0$ $(\gamma = 1, \cdots, p)$ and $\Psi^*_\nu = 0$ $(\nu = 1, \cdots, p^*)$ relative to the differential equations $\Phi_a = 0$, and denote by N_0 and P_0, respectively, the number of negative and positive zeros of the determinant $D_0(\rho)$ defined above. Then $\lambda^*_{s+N_0+q-p^*} \geq \lambda_s$, $\lambda_{s+P_0+q-p} \geq \lambda^*_s$ $(s = 1, 2, \cdots)$.*

COROLLARY. *Under the hypotheses of Theorem 5.5, for every L we have*

$$V_L - P_0 - (q - p) \leq V^*_L \leq V_L + N_0 + (q - p^*),$$
$$W_L - P_0 - (q - p) \leq W^*_L \leq W_L + N_0 + (q - p^*);$$

moreover, $| V[\Delta] - V^*[\Delta] | \leq N_0 + P_0 + 2q - p - p^*$ for every finite sub-interval Δ of the λ-axis, and $N_0 + P_0 + 2q - p - p^* \leq 2n - r$.

The last conclusion of this corollary is a ready consequence of the inequalities $N_0 + P_0 \leq 2n - q - r$, $q \leq p + p^*$.

6. Oscillation theorems. Suppose that $\omega(x, \eta, \eta')$, $\Phi_a(x, \eta, \eta')$ $(\alpha = 1, \cdots, m)$, $M_{ij}(x, t)$ are given and satisfy on $a \leq x \leq b$ hypotheses (H_1) and (H_2); moreover, that the end-form Q depends only upon $\eta_i(a)$, that is, $2Q \equiv Q_{ia;\,ja}\eta_i(a)\eta_j(a)$. Let $\| \mathfrak{P}_{i\rho} \|$ $(\rho = 1, \cdots, q \leq n;\ i = 1, \cdots, n)$ be a constant matrix of rank q, and denote by P the linear sub-space

$$P\colon\ x = a,\ \eta_{ia}\mathfrak{P}_{i\rho} = 0 \qquad (\rho = 1, \cdots, q) \tag{6.1}$$

in $(n + 1)$-dimensional (x, η_{ia}) space. For a given c on $a < c \leq b$, let $B(P\colon c)$ denote the normal boundary problem determined by ω, Φ_a, $M_{ij}(x, t)$, $Q[\eta(a)]$, and the end conditions

$$(6.2)\qquad \eta_i(a)\mathfrak{P}_{i\rho}=0,\ \eta_i(c)=0 \qquad (\rho=1,\cdots,q;\ i=1,\cdots,n).$$

Corresponding to a value c on $a<c\leq b$, denote by r_c the order of anormality on ac, and by $\eta_i=\eta_{ih}(x)\equiv 0$, $\zeta_i=\zeta_{ih}(x)$ $(h=1,\cdots,r_c)$ a linearly independent set of solutions of the integro-differential equations of $B(P:c)$ on ac. If the matrix $\|\mathfrak{P}_{i\rho};\zeta_{ih}(a)\|$ $(\rho=1,\cdots,q;h=1,\cdots,r_c;i=1,\cdots,n)$ has rank $q+r_c-k_c$, then $0\leq k_c\leq q$ and there are values $\rho_1,\cdots,\rho_{l(c)}$ $(l(c)=q-k_c)$ such that $1=\rho_1<\rho_2<\cdots<\rho_{l(c)}\leq q$ and the auxiliary end conditions $\Psi_\gamma=0$ $(\gamma=1,\cdots,p(c)=n+q-k_c)$ of $B(P:c)$ may be written

$$(6.3)\qquad \eta_i(a)\mathfrak{P}_{iv}=0,\ \eta_i(c)=0 \qquad (v=\rho_1,\rho_2,\cdots,\rho_{l(c)};\ i=1,\cdots,n).$$

It is to be remarked that the coefficients of the end conditions of $B(P:c)$ may be chosen the same for all values of c on an interval where r_c is constant.

The expression (1.1) for $B(P:c)$ will be denoted by $I[\eta:c]$. It is to be emphasized that the upper limit of the integrals in $I[\eta:c]$ is c. If $a<c<\mathbf{c}\leq b$, and $\eta_i=u_i(x)$ is an admissible arc for $B(P:c)$, then the arc $U_i=u_i(x)$ $(a\leq x\leq c)$, $U_i\equiv 0$ $(c\leq x\leq \mathbf{c})$ is admissible for $B(P:\mathbf{c})$ and $I[U:\mathbf{c}]=I[U:c]=I[u:c]$.

A value c will be called a *focal point of P on the interval ab relative to the system*

$$(6.4)\qquad L_i[\eta,\zeta]=0,\qquad L_{n+i}[\eta,\zeta|\lambda]=0$$

for $\lambda=\lambda_0$ *if:* (i) *$\lambda=\lambda_0$ is a characteristic value of $B(P:c)$*; (ii) *there is at least one corresponding characteristic solution $\eta_i=y_i(x)$, $\zeta_i=z_i(x)$ such that for arbitrary ϵ satisfying $0<\epsilon<b-c$ there is no corresponding set of functions $\zeta_i(x)$ forming with the arc $\eta_i=y_i(x)$ on ac, $\eta_i\equiv 0$ on $c\leq x\leq c+\epsilon$ a characteristic solution of $B(P:c+\epsilon)$ for $\lambda=\lambda_0$.* If $a<c<b$ and c is a focal point of P on ab relative to (6.4) for $\lambda=\lambda_0$, then c is a focal point of P on every interval ab' $(c<b'\leq b)$ relative to the same system. On the other hand, for $c=b$ condition (ii) of the above definition is satisfied vacuously, and b is a focal point of P on ab relative to the system (6.4) for $\lambda=\lambda_0$ whenever λ_0 is a characteristic value of $B(P:b)$. Consequently, in general $x=b$ may be a focal point of P on ab relative to (6.4) for $\lambda=\lambda_0$ and not be a focal point of P on ab'' $(b''>b)$ relative to this same system. If the integro-differential system (6.4) is normal on every sub-interval of an extension ab'' $(b''>b)$, then a point c of $a<c\leq b$ is a focal point on the interval ab'' relative to (6.4) for $\lambda=\lambda_0$ if and only if λ_0 is a characteristic value of $B(P:c)$. If y_i,z_i is a solution of $B(P:c)$ satisfying the above condition (ii), we shall say that $y\equiv[y_i(x)]$ is an arc *determining $x=c$ as*

a focal point of P. The number of such solutions of $B(P:c)$ which are linearly independent on ac will be termed the *index of c as a focal point.* If the linear space (6.1) reduces to the point $x=a$, $\eta_{ia}=0$, that is, if $q=n$, the corresponding focal points will be termed *conjugate points* of $x=a$.

We shall denote by $\{\lambda_s(c)\}$, $\eta_i = y_{is}(x:c)$, $\zeta_i = z_{is}(x:c)$ $(s=1,2,\cdots)$ the characteristic values and characteristic solutions of $B(P:c)$, supposed ordered and orthonormal in the sense of (4.7) and (4.8).

Lemma 6.1. *As* $c \to a$, $\lambda_1(c) \to +\infty$.

In view of Corollary 2 of Theorem 5.1 it is sufficient to prove this theorem for the associated differential problem $B^0(P:c)$. For $B^0(P:c)$ the desired result may be established by the method used by Hu ([9], Lemma 13.1), since his proof does not involve any assumption of normality.

Lemma 6.2. *Each of the characteristic values* $\lambda_s(c)$ *of* $B(P:c)$ *varies continuously with c, and increases from* $\lambda_s(b)$ *to* $+\infty$ *as c decreases from b to a.*

Suppose $a < c \leq \mathbf{c} \leq b$, and let $d_1, \cdots, d_s$ be constants such that the arc $\eta_i = y_{i1}(x:c)d_1 + \cdots + y_{is}(x:c)d_s$ on ac, $\eta_i \equiv 0$ on $c\mathbf{c}$ satisfies the relations $1 = K[\eta:c] = K[\eta:\mathbf{c}]$; $K[y_\kappa(x:\mathbf{c}), \eta:\mathbf{c}] = 0$, $(\kappa = 1,\cdots, s-1)$. The arc η thus defined is admissible for both $B(P:c)$ and $B(P:\mathbf{c})$. By Theorem 4.3 and the minimizing property of $\lambda_s(\mathbf{c})$ we have

$$\lambda_s(\mathbf{c}) \leq I[\eta:\mathbf{c}] = I[\eta:c] \leq \lambda_s(c),$$

that is, $\lambda_s(c)$ is a monotone decreasing function on $a < c \leq b$.

We shall now prove that $\lambda_s(c)$ is continuous on $a < c \leq b$. Suppose $a < \mathbf{c} < b$, and denote by $\boldsymbol{\Psi}_\gamma[\eta(a), \eta(\mathbf{c})] = 0$ $(\gamma = 1, \cdots, p(\mathbf{c}))$ the corresponding auxiliary boundary conditions (6.3) for $B(P:\mathbf{c})$. Let $\epsilon_1 > 0$ be such that for $a < \mathbf{c} - \epsilon_1 \leq c \leq \mathbf{c} + \epsilon_1 < b$ hypothesis (H_3) is satisfied by the end conditions $\boldsymbol{\Psi}_\gamma[\eta(a), \eta(c)] = 0$ $(\gamma = 1, \cdots, p(\mathbf{c}))$, and, moreover, ϵ_1 is such that r_c is constant on $\mathbf{c} - \epsilon_1 \leq c \leq \mathbf{c}$. For each c on this interval let $\mathbf{B}(P:c)$ denote the boundary problem determined by $I[\eta:c]$, Φ_a, $\boldsymbol{\Psi}_\gamma$. The ordered characteristic values and orthonormal characteristic solutions of $\mathbf{B}(P:c)$ will be denoted by $\boldsymbol{\lambda}_s(c)$, $\mathbf{y}_{is}(x:c)$, $\mathbf{z}_{is}(x:c)$ $(s = 1, 2, \cdots)$. Since the order of anormality on ac is constant for $\mathbf{c} - \epsilon_1 \leq c \leq \mathbf{c}$, for such values of c the problem $\mathbf{B}(P:c)$ is identical with $B(P:c)$. If r_c is also constant on $\mathbf{c} \leq c \leq \mathbf{c} + \epsilon_1$, $\mathbf{B}(P:c)$ is also identical with $B(P:c)$ on this interval. If, however, r_c has a discontinuity at $\mathbf{c}$, then $B(P:c)$ is a sub-problem of $\mathbf{B}(P:c)$ on $\mathbf{c} \leq c \leq \mathbf{c} + \epsilon_1$ and, by Theorem 5.4, $\lambda_s(c) \geq \boldsymbol{\lambda}_s(c)$ $(s = 1, 2, \cdots)$.

It will now be proved that each $\boldsymbol{\lambda}_s(c)$ is continuous at $c = \boldsymbol{c}$. Let $\boldsymbol{s}_\sigma[\eta, \zeta : c] = 0$ $(\sigma = 1, \cdots, 2n)$ denote the boundary conditions of $\boldsymbol{B}(P : c)$. By the Corollary to Lemma 3. 2 there is a λ_0 such that for c on $|c - \boldsymbol{c}| \leqq \epsilon_1$ the differential boundary problem

(6. 5) $\boldsymbol{L}_i[\eta,\zeta] \equiv L_i[\eta,\zeta] = 0,\ \boldsymbol{L}_{n+i}[\eta,\zeta] \equiv \zeta'_i - \Omega_{\eta_i} + \lambda_0 \eta_i = 0,\ \boldsymbol{s}_\sigma[\eta,\zeta : c] = 0$

is incompatible. Suppose $\boldsymbol{c} - \epsilon_1 \leq c \leq c' \leq \boldsymbol{c} + \epsilon_1$, and consider the non-homogeneous system

(6. 6) $\boldsymbol{L}_i[\eta, \zeta] = 0,\ \boldsymbol{L}_{n+i}[\eta, \zeta] = h_{if}(x : c'),\ \boldsymbol{s}_\sigma[\eta, \zeta : c] = 0$ $(f = 1, \cdots, s)$,

where

$$h_{if}(x : c') = [\lambda_0 - \boldsymbol{\lambda}_f(c')]\boldsymbol{y}_{if}(x : c') + \int_a^{c'} M_{ij}(x, t)\boldsymbol{y}_{if}(t : c')\,dt.$$

If $\eta_{if}(x : c, c')$, $\zeta_{if}(x : c, c')$ is the solution of (6. 6), we have

$$\eta_{if}(x : c', c') = \boldsymbol{y}_{if}(x : c'), \qquad \zeta_{if}(x : c', c') = \boldsymbol{z}_{if}(x : c').$$

For arbitrary constants d_f $(f = 1, \cdots, s)$, set

$$I[d : c, c'] = I[\eta_f(x : c, c')d_f : c], \qquad K[d : c, c'] = K[\eta_f(x : c, c')d_f : c].$$

Using the explicit form of $\eta_{if}(x : c, c')$, $\zeta_{if}(x : c, c')$ as given by the Green's matrix of (6. 5), together with Lemma 3. 3, it follows that the coefficients of the quadratic forms $I[d : c, c']$, $K[d : c, c']$ in the variables d_f are continuous functions of c, c' uniformly on $\boldsymbol{c} - \epsilon_1 \leq c \leq c' \leq \boldsymbol{c} + \epsilon_1$. Since

$$I[d : c', c'] = \sum_{f=1}^{s} \boldsymbol{\lambda}_f(c') d_f^2, \qquad K[d : c', c'] = \sum_{f=1}^{s} d_f^2,$$

for a given ϵ $(0 < \epsilon < 1)$ there is a $\delta_\epsilon < \epsilon$ such that if $\boldsymbol{c} - \epsilon_1 \leq c \leq c' \leq \boldsymbol{c} + \epsilon_1$, $c' - c \leq \delta_\epsilon$, then

$$K[d : c, c'] \geq (1 - \epsilon) d_f d_f,\ I[d : c, c'] \leq [\boldsymbol{\lambda}_s(c') + \epsilon] \cdot K[d : c, c'].$$

This implies, in particular, that the functions $\eta_{if}(x : c, c')$ are linearly independent on ac. Now the constants d_f may be chosen so that the arc $\eta_i = \eta_{if}(x : c, c')d_f$ satisfies the relations $K[\boldsymbol{y}_\kappa(x : c)\,;\eta : c] = 0$, $(\kappa = 1, \cdots, s-1)$, $K[\eta : c] = 1$, and for such a choice we have $\boldsymbol{\lambda}_s(c) \leq I[\eta : c] \leq \boldsymbol{\lambda}_s(c') + \epsilon$. In particular, if $0 \leq \delta \leq \delta_\epsilon$ we have

$$\lambda_s(\boldsymbol{c}) + \epsilon = \boldsymbol{\lambda}_s(\boldsymbol{c}) + \epsilon \geq \boldsymbol{\lambda}_s(\boldsymbol{c} - \delta) = \lambda_s(\boldsymbol{c} - \delta) \geq \lambda_s(\boldsymbol{c}),$$
$$\lambda_s(\boldsymbol{c}) - \epsilon = \boldsymbol{\lambda}_s(\boldsymbol{c}) - \epsilon \leq \boldsymbol{\lambda}_s(\boldsymbol{c} + \delta) \leqq \lambda_s(\boldsymbol{c} + \delta) \leq \lambda_s(\boldsymbol{c}),$$

that is, $|\lambda_s(\boldsymbol{c} \pm \delta) - \lambda_s(\boldsymbol{c})| \leq \epsilon$. Since ϵ may be chosen arbitrarily small we have proved that $\lambda_s(c)$ is continuous at $c = \boldsymbol{c}$, whenever $a < \boldsymbol{c} < b$. The continuity of $\lambda_s(c)$ at $c = b$ is proved by the same method, where in the above formulas $\boldsymbol{c} = b$, and the comparison values c are restricted by the inequality $b - \epsilon_1 \leq c \leq b$.

THEOREM 6. 1. *The number of focal points of P on $a < x < b$ relative to the system* (6. 4) *for a given value $\lambda = \lambda^*$ is equal to the number of characteristic values $\lambda_s(b)$ of $B(P:b)$ which are less than λ^*.*

Suppose that there are exactly s characteristic values $\lambda_f(b)$ $(f = 1, \cdots, s)$ of $B(P:b)$ which are less than λ^*, and for each value of f let $x = c_f$ denote the last value on $a < x < b$ for which $\lambda_f(x) = \lambda^*$. In view of Lemmas 6. 1 and 6. 2 these values c_f are well-defined. It will first be proved that if $a < c < b$ and $x = c$ is a focal point of P relative to (6. 4) for $\lambda = \lambda^*$, then c is one of the values c_f. For suppose c is such a focal point which is distinct from the values c_f, and denote by $y \equiv [y_i(x)]$ an arc determining c as a focal point of P. We may without loss of generality assume $K[y:c] = 1$. Let N denote the largest integer such that $\lambda_N(c) < \lambda^*$. Then $\lambda_{N+1}(c) = \lambda^*$, and since c is distinct from the values c_f there exists an $\epsilon > 0$ such that $\lambda_{N+1}(x) \equiv \lambda^*$ on $c \leq x \leq c + \epsilon$. Now consider an arc η of the form

$$\eta_i = y_{i1}(x:c)d_1 + \cdots + y_{iN}(x:c)d_N + y_i(x)d_{N+1} \quad (a \leq x \leq c),$$

and $\eta_i \equiv 0$ on $c \leq x \leq c + \epsilon$, where the constants $d_1, \cdots, d_{N+1}$ are such that

$$(6.7) \quad 1 = K[\eta:c+\epsilon] = K[\eta:c], \quad K[y_\kappa(x:c+\epsilon);\eta:c+\epsilon] = 0 \quad (\kappa = 1, \cdots, N).$$

In view of the minimizing property of $\lambda_{N+1}(c+\epsilon)$ we have $I[\eta:c+\epsilon] \geq \lambda_{N+1}(c+\epsilon)$. By Theorem 4. 3, however,

$$I[\eta:c+\epsilon] = I[\eta:c] \leq \lambda_{N+1}(c) = \lambda_{N+1}(c+\epsilon).$$

Hence $I[\eta:c+\epsilon] = \lambda_{N+1}(c+\epsilon)$, $0 = d_1 = \cdots = d_N$ by (4. 9), and in view of (6. 7) and Theorem 4. 4 there exist functions ζ_i such that η_i, ζ_i is a characteristic solution of $B(P:c+\epsilon)$. This, however, contradicts the assumption that y is an arc determining c as a focal point of P. We have established, therefore, that if c is a focal point of P on $a < x < b$ relative to (6. 4) for $\lambda = \lambda^*$, then c is one of the values c_f $(f = 1, \cdots, s)$ defined above.

It will now be proved that each c_f $(f = 1, \cdots, s)$ is a focal point of P relative to (6. 4) for $\lambda = \lambda^*$, and that the index of c_f is equal to the number of characteristic values $\lambda_\kappa(c_f)$ for which $\kappa \leq s$ and $\lambda_\kappa(c_f) = \lambda^*$. It is obviously sufficient to prove this result for the largest value c_s, since each c_f may be considered the largest such value on a suitable subinterval of ab. Suppose that $\lambda^* = \lambda_s(c_s) = \lambda_{s-1}(c_s) = \cdots = \lambda_{s-g+1}(c_s)$, and either $s - g = 0$ or $\lambda_{s-g}(c_s) < \lambda^*$. It is also quite possible that some of the values $\lambda_\kappa(c_s)$ with $\kappa > s$ are also equal to λ^*; if this is the case, however, the corresponding $\lambda_\kappa(c)$ is constant on $c_s \leq c \leq b$. We shall suppose, for definiteness, that there do

exist values $\lambda_\kappa(c_s) = \lambda^*$ with $\kappa > s$, and shall give the proof of the theorem in this more complicated case. If $\lambda_{s+1}(c_s) > \lambda^*$ the theorem is proved in a similar manner, but with obvious simplifications. If

$$\lambda^* = \lambda_{s+1}(c_s) = \cdots = \lambda_{s+t}(c_s), \qquad \lambda_{s+t+1}(c_s) > \lambda^*,$$

set

$$\eta_i = y_{i1}(x:c_s)d_1 + \cdots + y_{i\,s+t}(x:c_s)d_{s+t}$$

on ac_s, $\eta_i \equiv 0$ on c_sb, where the constants $d_1, \cdots, d_{s+t}$ are chosen so that

$$(6.8) \quad K[y_f(x:b);\eta:b] = 0 \quad (f = 1, \cdots, s),$$
$$1 = d_1^2 + \cdots + d^2_{s+t} = K[\eta:c_s] = K[\eta:b].$$

From the minimizing property of $\lambda_{s+1}(b)$ we have $I[\eta:b] \geq \lambda_{s+1}(b) = \lambda^*$. On the other hand, by Theorem 4.3, $I[\eta:b] = I[\eta:c_s] \leq \lambda_{s+t}(c_s) = \lambda^*$. By Theorem 4.4 there are functions ζ_i such that η_i, ζ_i is a characteristic solution of $B(P:b)$ for $\lambda = \lambda^*$. Since the functions $y_{i\kappa}(x:c_s)$ $(\kappa = 1, \cdots, s+t)$ are linearly independent on ac_s, there are seen to exist at least t linearly independent solutions η_{il}, ζ_{il} $(l = s+1, \cdots, s+t)$ of $B(P:b)$ for $\lambda = \lambda^*$ with $\eta_{il} \equiv 0$ on c_sb. Now η_{il}, ζ_{il} $(l = s+1, \cdots, s+t)$ are linearly independent solutions of $B(P:c_s)$ for $\lambda = \lambda^*$ such that no arc $\eta_{il}(x)$ is an arc determining c_s as a focal point of P. Consequently, the index of c_s as a focal point is at most g. On the other hand, there are g solutions $\eta_{i\kappa}, \zeta_{i\kappa}$ $(\kappa = s-g+1, \cdots, s)$ of $B(P:c_s)$ for $\lambda = \lambda^*$ satisfying the relations $K[\eta_\kappa;\eta_\mu:c_s] = \delta_{\kappa\mu}$, $K[\eta_l;\eta_\kappa:c_s] = 0$ $(\kappa, \mu = s-g+1, \cdots, s;$ $l = s+1, \cdots, s+t)$. Each of the arcs determines c_s as a focal point of P, since otherwise there would exist an $\epsilon > 0$ such that for every c on $c_s \leq c \leq c_s + \epsilon$ the index of λ^* as a characteristic value of $B(P:c)$ is greater than t. This, however, is impossible in view of the definition of c_s and since $\lambda_{s+t+1}(c)$ remains greater than λ^* for c in a neighborhood of c_s. Hence the index of c_s as a focal point of P relative to (6.4) for $\lambda = \lambda^*$ is equal to g.

Corresponding to a given problem B satisfying hypotheses (H), the normal boundary problem determined by the $I[\eta]$, $\Phi_a = 0$ belonging to B, together with the end conditions $\eta_i(a) = 0$, $\eta_i(b) = 0$, will be termed the *associated null end point problem.* Clearly, the associated null end point problem is a sub-problem of B of dimension at most $2n - p$. In view of Theorems 5.4 and 6.1 we have immediately:

THEOREM 6.2. *If for a given problem B satisfying hypotheses (H) the associated null end point problem is a sub-problem of B of dimension d, then for a given L the number of conjugate points of $x = a$ relative to the equations*

(6.4) *of B for $\lambda = L$, and located on the open interval $a < x < b$, is at least $V_L - d$ and at most V_L.*

7. A problem non-linear in the characteristic parameter. We shall now consider a boundary problem involving a parameter λ in such a manner that for every value of λ there is defined a self-adjoint integro-differential system of the sort treated in the preceding sections. It will be supposed that the functions M_{ij} and the coefficients of the quadratic forms ω, Q involve a real parameter λ, and are continuous in their arguments for $a \leq x,\ t \leq b$, $\mathfrak{C}: \mathfrak{C}_1 < \lambda < \mathfrak{C}_2$; the coefficients of Φ_α and Ψ_γ are supposed to be independent of λ. Moreover, it is supposed that for each value of λ on $\mathfrak{C}$ hypotheses (H) of Section 2 are satisfied. For brevity, these conditions will be denoted by (H^λ). The coefficients A_{ij}, B_{ij}, $l_{\alpha j}$, $m_{\alpha j}$ occurring in the solution (2.7) of the system (2.6) now depend upon λ, and we may write the canonical form of our integro-differential system as

$$
(7.1)\quad
\begin{aligned}
&L_i[\eta, \zeta : \lambda] \equiv \eta'_i - A_{ij}(x:\lambda)\eta_j - B_{ij}(x:\lambda)\zeta_j = 0,\\
&L_{n+i}[\eta, \zeta : \lambda] \equiv \zeta'_i - C_{ij}(x:\lambda)\eta_j\\
&\qquad\qquad + A_{ji}(x:\lambda)\zeta_j - \int_a^b M_{ij}(x, t:\lambda)\eta_j(t)\,dt = 0,\\
&s_\sigma[\eta, \zeta : \lambda] = 0, \qquad\qquad (i, j = 1, \cdots, n;\ \sigma = 1, \cdots, 2n).
\end{aligned}
$$

The boundary conditions $s_\sigma = 0$ have the explicit form (2.5′), where it is to be remembered that the coefficients of Q depend upon λ. A value λ will be said to be a characteristic value of this boundary problem if there exist functions $\eta_i(x), \zeta_i(x)$ of class C^1, not all identically zero on ab and satisfying the system (7.1). The expression (1.1) for this new problem will be denoted by $I[\eta : \lambda]$.

In order to discuss the characteristic values of (7.1) we consider the auxiliary boundary problem

$$
(7.2)\quad L_i[\eta, \zeta : \lambda] = 0, \quad L_{n+i}[\eta, \zeta : \lambda] + \mu\eta_i = 0, \quad s_\sigma[\eta, \zeta : \lambda] = 0,
$$

which involves the parameter μ linearly. In view of the above hypotheses and the results of Section 4, for each value of λ on $\mathfrak{C}$ there are infinitely many characteristic values $\mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots$ of (7.2). The corresponding characteristic solutions $\eta_i = y_{is}(x:\lambda)$, $\zeta_i = z_{is}(x;\lambda)$ are supposed orthonormal in the sense that

$$
\int_a^b y_{is}(x:\lambda)y_{it}(x:\lambda)\,dx = \delta_{st} \qquad (s, t = 1, 2, \cdots).
$$

The class of admissible arcs η such that $K[\eta] = 1$, $K[y_\kappa(x:\lambda); \eta] = 0$

$(\kappa = 1, \cdots, s-1)$ will be denoted by $\mathfrak{S}_s(\lambda)$. It follows from Section 4 that $\mu_s(\lambda)$ is the minimum of $I[\eta:\lambda]$ in the class $\mathfrak{S}_s(\lambda)$. Clearly a value λ is a characteristic value of (7.1) if and only if there is an integer s such that $\mu_s(\lambda) = 0$; moreover, the index of λ as a characteristic value of (7.1) is equal to the index of $\mu = 0$ as a characteristic value of the corresponding system (7.2).

THEOREM 7.1. *If hypotheses* (H^λ) *are satisfied, the characteristic values* $\mu_s(\lambda)$ *of* (7.2) *are continuous functions of* λ *on* $\mathfrak{C}$.

The proof of this lemma uses Lemma 3.2. In view of the continuity of the coefficients of $I[\eta:\lambda]$ we have that for a given bounded and closed subinterval $\mathfrak{C}_0: \lambda_0 \leq \lambda \leq \lambda'_0$ of $\mathfrak{C}$ there exist positive constants $k(\mathfrak{C}_0)$, $l(\mathfrak{C}_0)$, $m(\mathfrak{C}_0)$ such that for λ on $\mathfrak{C}_0$ the inequalities

$$(7.3) \qquad \int_a^b \eta'_i\eta'_i dx \leq m(\mathfrak{C}_0)I[\eta:\lambda] + l(\mathfrak{C}_0)K[\eta],$$

$$(7.4) \quad I[\eta:\lambda] \leq k(\mathfrak{C}_0)[\eta_i(a)\eta_i(a) + \eta_i(b)\eta_i(b) + \int_a^b (\eta'_i\eta'_i + \eta_i\eta_i)dx],$$

hold for arbitrary differentially admissible arcs η. In view of (7.4) and Theorem 5.1 there are constants $l_s(\mathfrak{C}_0)$ $(s = 1, 2, \cdots)$ such that $\mu_s(\lambda) \leq l_s(\mathfrak{C}_0)$ $(s = 1, 2, \cdots)$ for λ on $\mathfrak{C}_0$.

Now let λ, λ' be values on $\mathfrak{C}_0$ and choose constants $d_1, \cdots, d_s$ such that the arc $u_i = y_{i1}(x:\lambda')d_1 + \cdots + y_{is}(x:\lambda')d_s$ belongs to $\mathfrak{S}_s(\lambda)$. We then have $\mu_s(\lambda) \leq I[u:\lambda]$, $I[u:\lambda'] \leq \mu_s(\lambda')$. Hence

$$\mu_s(\lambda) - \mu_s(\lambda') \leq I[u:\lambda] - I[u:\lambda'] \leq \epsilon_1(\lambda,\lambda')[u_i(a)u_i(a) + u_i(b)u_i(b)] + \epsilon_2(\lambda,\lambda')\int_a^b (u'_iu'_i + u_iu_i)dx,$$

where $\epsilon_1(\lambda,\lambda')$, $\epsilon_2(\lambda,\lambda')$ are independent of the particular arc u, are symmetric in (λ,λ'), and tend to zero with $|\lambda - \lambda'|$ uniformly for λ, λ' on $\mathfrak{C}_0$. This inequality is a consequence of the continuity of the coefficients of $I[\eta:\lambda]$ as functions of λ. Let $\phi(x) = 2(x-a)/(b-a) - 1$. Then $\phi(a) = -1$, $\phi(b) = 1$, $|\phi'| = 2/(b-a)$, and $|\phi| \leq 1$ on ab. Now

$$\begin{aligned} |u_i(a)u_i(a) + u_i(b)u_i(b)| &\leq \int_a^b |2\phi u_iu'_i + \phi'u_iu_i|\, dx, \\ &\leq \int_a^b [|\phi|\, u'_iu'_i + (|\phi| + |\phi'|)u_iu_i]dx, \\ &\leq \int_a^b u'_iu'_i dx + M_1, \end{aligned}$$

where $M_1 = 1 + 2/(b-a)$. Hence, for $\epsilon_3 = \epsilon_1 + \epsilon_2$, $\epsilon_4 = M_1\epsilon_1 + \epsilon_2$ we have,

$$
\begin{aligned}
\mu_s(\lambda) - \mu_s(\lambda') &\leq \epsilon_3(\lambda, \lambda') \int_a^b u'_i u'_i dx + \epsilon_4(\lambda, \lambda'), \\
&\leq \epsilon_3(\lambda, \lambda')[m(\mathfrak{C}_0) I[u : \lambda'] + l(\mathfrak{C}_0)] + \epsilon_4(\lambda, \lambda'), \\
&\leq \epsilon_3(\lambda, \lambda')[m(\mathfrak{C}_0)\, l_s(\mathfrak{C}_0) + l(\mathfrak{C}_0)] + \epsilon_4(\lambda, \lambda'),
\end{aligned} \tag{7.5}
$$

since $I[u : \lambda'] \leq \mu_s(\lambda') \leq l_s(\mathfrak{C}_0)$. As ϵ_3, ϵ_4 are symmetric in (λ, λ'), it follows that $|\mu_s(\lambda) - \mu_s(\lambda')|$ does not exceed the right-hand member of inequality (7. 5), and the continuity of $\mu_s(\lambda)$ on $\mathfrak{C}_0$ is immediate.

We shall now consider a system which satisfies the following additional hypotheses:

$(H_4{}^\lambda)$ For λ sufficiently near $\mathfrak{C}_1$, $I[\eta : \lambda] > 0$ for all non-identically vanishing admissible arcs η.

$(H_5{}^\lambda)$ For each integer s, if $\mu_s(\lambda) = 0$ for $\lambda = \lambda'$, then $\mu_s(\lambda) < 0$ for $\lambda > \lambda'$.

Suppose that B is the above defined problem involving $\omega(x, \eta, \eta' : \lambda)$, $Q[\eta(a), \eta(b) : \lambda]$, $M_{ij}(x, t : \lambda)$, $\phi_\alpha(x, \eta, \eta')$, $\Psi_\gamma[\eta(a), \eta(b)]$, that B^* is a second problem involving $\omega^*(x, \eta, \eta' : \lambda)$, $Q^*[\eta(a), \eta(b) : \lambda]$, $M^*_{ij}(x, t : \lambda)$, $\Phi_\alpha(x, \eta, \eta')$, $\Psi^*_\nu[\eta(a), \eta(b)]$, and that B and B^* each satisfy hypotheses (H^λ), $(H_4{}^\lambda)$, $(H_5{}^\lambda)$. Let $N(L)$ and $P(L)$ denote, respectively, the number of negative and positive zeros of $D(\rho : L)$ defined by (5. 1) in terms of the difference end-form $d[\eta(a), \eta(b) : L] = Q^*[\eta(a), \eta(b) : L] - Q[\eta(a), \eta(b) : L]$. Similarly, the N_0, P_0 occurring in Theorem 5. 5 are now replaced by $N_0(L)$, $P_0(L)$. By considering the curves $\mu = \mu_s(\lambda)$ it is readily seen that the conclusions of Theorem 5. 1, Theorem 5. 4 and its corollary, together with the inequalities involving V_L, W_L, V^*_L, W^*_L occurring in the corollaries to Theorems 5. 3 and 5. 5, hold for the above defined problems B and B^*, with the understanding that the $I[\eta]$, $I^*[\eta]$, N, P, N_0, P_0 occurring in the statements of these results are now replaced by $I[\eta : \lambda]$, $I^*[\eta : \lambda]$, $N(L)$, $P(L)$, $N_0(L)$, $P_0(L)$. Corresponding to the corollary of Theorem 5. 2 we now have under the hypotheses corresponding to those of Theorem 5. 2 that if λ_s exists for B and if h_s denote the number of characteristic values of $D(B, B^*)$ less than λ_s, then $\lambda^*_{s+h_s}$, if it exists, is not less than λ_s. It is also readily seen that Theorems 6. 1 and 6. 2 hold for a problem B satisfying the above hypotheses. It is to be noted that for the above theorems we have not supposed that B and B^* have infinitely many characteristic values.

Applying the results of Theorems 4. 2 and 4. 3 to the problem (7. 2), we see that $(H_5{}^\lambda)$ holds if $I[\eta : \lambda]$ is a proper monotone decreasing function of λ for all non-identically vanishing admissible arcs η. This latter condition is in turn assured if the coefficients of the functions occurring in $I[\eta : \lambda]$ have suitable monotone properties as functions of λ. Such conditions are assumed

in the classical Sturmian theory for a second-order differential equation, and have also been used by Morse [16].

We shall now consider a problem (7.1) which satisfies the following condition.

(H_6^λ) There is a function $g(\lambda', \lambda)$ defined for $\mathfrak{C}_1 < \lambda', \lambda < \mathfrak{C}_2$, $\lambda > \lambda'$ which is positive and such that if $\lambda > \lambda'$ then

$$I[\eta : \lambda] < g(\lambda', \lambda) I[\eta : \lambda']$$

for all non-identically vanishing admissible arcs η.

THEOREM 7.2. *If a problem satisfies hypotheses* (H^λ), (H_4^λ), (H_6^λ), *then* (H_5^λ) *is also satisfied.*

For suppose that for a value λ' there is an integer s such that $\mu_s(\lambda') = 0$. For arbitrary constants d_κ ($\kappa = 1, \cdots, s$) the arc $\eta_i = \sum_{\kappa=1}^{s} y_{i\kappa}(x : \lambda') d_\kappa$ is such that $I[\eta : \lambda'] \leq \mu_s(\lambda') = 0$. By ($H_6^\lambda$), for $\lambda > \lambda'$ each of these arcs is such that $I[\eta : \lambda] < g(\lambda', \lambda) I[\eta : \lambda'] \leq 0$. It then follows from the extremizing properties of the characteristic values of (7.2) that $\mu_s(\lambda) < 0$ for $\lambda > \lambda'$.

Now consider a problem of the form formulated in Section 2 which satisfies in addition to hypotheses (H) the condition that $I[\eta] > 0$ for all non-identically vanishing admissible arcs λ. Let $\tilde{B}$ denote the boundary problem determined by

$$I[\eta : \lambda] = I[\eta] - \lambda \left\{ \mathfrak{Q}[\eta(a), \eta(b)] + \int_a^b K_{ij}(x) \eta_i \eta_j dx + \int_a^b \int_a^b N_{ij}(x, t) \eta_i(x) \eta_j(t) dx dt \right\}, \tag{7.6}$$

together with the differential equations $\Phi_\alpha = 0$ and the end-conditions $\Psi_\gamma = 0$ belonging to the first problem. It is to be understood that $\mathfrak{Q}$ is a quadratic form in $[\eta_i(a), \eta_i(b)]$, and that $\| K_{ij}(x) \|$, $\| N_{ij}(x, t) \|$ are matrices whose elements are continuous with $K_{ij}(x) = K_{ji}(x)$, $N_{ij}(x, t) = N_{ji}(t, x)$. Clearly this problem $\tilde{B}$ satisfies (H^λ) and (H_4^λ) on $0 < \lambda < +\infty$. It also satisfies (H_6^λ) since for $\lambda > \lambda' > 0$ and arbitrary admissible arcs η,

$$I[\eta : \lambda] - I[\eta : \lambda'] = [(\lambda - \lambda')/\lambda'] \{ I[\eta : \lambda'] - I[\eta] \}.$$

Using the notation of Section 2, this system may be written as

$$\begin{aligned} &L_i[\eta, \zeta] = 0, \; L_{n+i}[\eta, \zeta | 0] + \lambda \left\{ K_{ij}(x) \eta_j + \int_a^b N_{ij}(x, t) \eta_j(t) dt \right\} = 0, \\ &s_\sigma[\eta, \zeta : \lambda] = 0, \end{aligned} \tag{7.7}$$

where the boundary conditions are linear in λ. The above results give oscillation and comparison theorems for the positive characteristic values and associated characteristic solutions of (7.6). To obtain corresponding results for the negative characteristic values, one need only replace the form $\mathfrak{Q}$ and the functions K_{ij}, N_{ij} by their negatives, and consider the resulting system, again for $0 < \lambda < +\infty$. In case the functions $M_{ij}(x, t)$ occurring in $I[\eta]$ and the functions $N_{ij}(x, t)$ are identically zero, system (7.7) reduces to the differential system considered by the author [18].

We shall now consider the question of the existence of infinitely many characteristic values of (7.1). Clearly such a system which satisfies (H^λ), $(H_4{}^\lambda)$, $(H_5{}^\lambda)$ will have infinitely many characteristic values if and only if for each integer s there is a value λ' such that $\mu_s(\lambda') = 0$. In view of the extremizing properties of the values $\mu_s(\lambda)$ we have that a system (7.1) satisfying the above hypotheses will have at least k characteristic values if and only if there are k admissible functions $\eta_{i\kappa}(x)$ $(\kappa = 1, \cdots, s)$ and a value λ' such that for arbitrary constants $(d_\kappa) \neq (0_\kappa)$ we have $I[\eta_\kappa d_\kappa : \lambda'] < 0$. In view of the analogue of the corollary to Theorem 5.5, an arbitrary problem (7.1) satisfying the above hypotheses will have an infinity of characteristic values if and only if there are infinitely many characteristic values of the normal problem determined by the null end-conditions $\eta_i(a) = 0 = \eta_i(b)$, and having $M_{ij}(x, t : \lambda)$, $\omega(x, \eta, \eta' : \lambda)$, $\Phi_\alpha(x, \eta, \eta')$ in common with the given problem. The following special criterion is readily proved.

THEOREM 7.3. *Suppose that for a problem* (7.1) *satisfying* (H^λ), $(H_4{}^\lambda)$, $(H_5{}^\lambda)$ *there are functions* $R(\lambda) > 0$, $P(\lambda)$ *on* $\mathfrak{C}$ *such that* $P(\lambda) \to +\infty$ *as* $\lambda \to \mathfrak{C}_2$, *and for arbitrary admissible arcs* η *vanishing at* $x = a$ *and* $x = b$ *we have*

$$(7.8) \qquad \int_a^b 2\omega(x, \eta, \eta' : \lambda)\, dx + \int_a^b \int_a^b M_{ij}(x, t : \lambda)\eta_i(x)\eta_j(t)\, dx dt \\ \leq \int_a^b R(\lambda)[\eta'_i\eta'_i - P(\lambda)\eta_i\eta_i]\, dx.$$

Then this problem has infinitely many real characteristic values on $\mathfrak{C}$.

It is to be noted that (7.8) is assured if there are functions $R(\lambda) > 0$, $P_1(\lambda)$, $P_2(\lambda)$ such that

$$(7.9) \qquad 2\omega(x, \eta, \pi : \lambda) \leq R(\lambda)[\pi_i\pi_i - P_2(\lambda)\eta_i\eta_i]$$

for arbitrary sets (η_i, π_i), the inequality

$$(7.10) \qquad \int_a^b \int_a^b M_{ij}(x, t : \lambda)\eta_i(x)\eta_j(t)\, dx dt \leq R(\lambda)P_1(\lambda) \int_a^b \eta_i\eta_i\, dx$$

holds for arbitrary continuous functions $\eta_i(x)$, and $P_1(\lambda) - P_2(\lambda) \to -\infty$ as $\lambda \to \mathfrak{C}_2$. Using the Schwarz integral inequality, it follows that (7.10) is certainly true if there is a function $P_3(\lambda) \geq 0$ such that $|M_{ij}(x, t:\lambda)| \leq R(\lambda)P_3(\lambda)/[n(b-a)]$ for $a \leq x, t \leq b$, λ on $\mathfrak{C}$, and $P_3(\lambda) - P_2(\lambda) \to \infty$ as $\lambda \to \mathfrak{C}_2$.

In conclusion, we shall consider the question of infinitely many characteristic values of the problem $\tilde{B}$ given by (7.7), where it is understood that the quantities $I[\eta]$, Q, Φ_α, Ψ_γ involved satisfy hypotheses (H) of Section 2, and $I[\eta] > 0$ for all non-identically vanishing admissible arcs η. For brevity, we write

$$(7.11) \qquad \mathfrak{N}[\eta] \equiv \int_a^b \int_a^b N_{ij}(x, t)\eta_i(x)\eta_j(t)\,dx dt,$$

$$\mathfrak{K}[\eta] = \mathfrak{Q}[\eta(a), \eta(b)] + \int_a^b K_{ij}\eta_i\eta_j dx + \mathfrak{N}[\eta].$$

The following theorem may be proved by the method used by the author ([19], pp. 846-848) in considering the corresponding result for a differential system.

THEOREM 7.4. *Suppose the problem $\tilde{B}$ satisfies the additional condition that $\mathfrak{K}[\eta] > 0$ (< 0) for arbitrary non-identically vanishing admissible arcs η. Then the characteristic values of $\tilde{B}$ are all positive (negative), and are infinite in number.*

THEOREM 7.5. *Suppose that the quantities $I[\eta]$, Q, Φ_α, Ψ_γ involved in the problem $\tilde{B}$ satisfy hypotheses (H), but not necessarily the condition that $I[\eta] > 0$ for all non-identically admissible arcs η. If, however, we also have: (i) the matrix $\| K_{ij} \|$ is positive definite on ab; (ii) $\mathfrak{Q}[\eta(a), \eta(b)] + \mathfrak{N}[\eta] \geq 0$ for arbitrary admissible arcs η, then $\tilde{B}$ has infinitely many positive and only a finite number of negative characteristic values.*

Since $\| K_{ij} \|$ is positive definite, we have by the Schwarz inequality that there is a constant l such that $| M[\eta] | \leq l \int_a^b K_{ij}\eta_i\eta_j dx$. It follows (see Morse [15], p. 533, and Reid [18], p. 786) that there is a positive constant λ_0 such that

$$0 < I[\eta] + \lambda_0 \int_a^b K_{ij}\eta_i\eta_j dx \leq I[\eta] + \lambda_0\mathfrak{K}[\eta]$$

for all non-identically vanishing admissible arcs η. If $I[\eta]$ is replaced by $I[\eta] + \lambda_0\mathfrak{K}[\eta]$, the modified problem is equivalent to the original problem by a linear change of parameter. By Theorem 7.4, however, the characteristic values of the modified problem are all positive and infinite in number. Hence

the original problem has infinitely many characteristic values of which only a finite number are negative.

One may also prove for $\tilde{B}$ a theorem analogous to Theorem 6.1 of Reid [18], but such a theorem will not be explicitly stated here.

8. Properties of integro-differential systems. In this section we shall state some significant properties of a general integro-differential system of the first order.

$$(8.1)\quad \begin{aligned} L_i[y] &\equiv y'_i - A_{ij}(x)y_j - \int_a^b C_{ij}(x,t)y_j(t)\,dt = 0, \\ s_i[y] &\equiv M_{ij}y_j(a) + N_{ij}y_j(b) = 0 \qquad (i,j = 1,\cdots,m). \end{aligned}$$

For brevity no proofs will be given, since these results may be established by a combination of the fundamental theorems of the Fredholm integral equation theory and the methods used by Bliss [2] in treating a differential boundary problem. It will be assumed that the functions $A_{ij}(x)$, $C_{ij}(x,t)$ are continuous on $a \leq x, t \leq b$, and that the constant matrix $\| M_{ij}; N_{ij} \|$ has rank m. Throughout this section the subscripts i, j, k, l will have the range $1,\cdots,m$. It is to be noted that the integro-differential system of Section 2 is of the form (8.1) for $m = 2n$ and $y \equiv [\eta_1,\cdots,\eta_n,\zeta_1,\cdots,\zeta_n]$.

Let $p_k = P_{kj}$, $q_k = Q_{kj}$ $(j = 1,\cdots,m)$ be linearly independent solutions of the equations $M_{ik}p_k - N_{ik}q_k = 0$ $(i = 1,\cdots,m)$. In accordance with the definition of Bliss [2] for a differential system, we say that the boundary conditions $t_i[z] \equiv z_j(a)P_{ji} + z_j(b)Q_{ji}$ $(i = 1,\cdots,m)$ are adjoint to the boundary conditions $s_i[y] = 0$. Furthermore, the integro-differential system

$$(8.2)\quad M_i[z] \equiv z'_i + z_jA_{ji}(x) + \int_a^b z_j(t)C_{ji}(t,x)\,dt = 0, \quad t_i[z] = 0$$

will be said to be the system adjoint to (8.1). It is to be noted that if $y_i(x)$ and $z_i(x)$ are solutions of $L_i[y] = 0$ and $M_i[z] = 0$, respectively, then $y_i(x)z_i(x)$ is constant on ab.

Let $\| M^*_{ij} \|$, $\| N^*_{ij} \|$, $\| P^*_{ij} \|$, $\| Q^*_{ij} \|$ be matrices such that the matrices of $2m$ rows and columns

$$\begin{Vmatrix} M_{ij} & N_{ij} \\ M^*_{ij} & N^*_{ij} \end{Vmatrix}, \qquad \begin{Vmatrix} -P^*_{ij} & -P_{ij} \\ Q^*_{ij} & Q_{ij} \end{Vmatrix}$$

are reciprocals. Then the expressions

$$s^*_i[y] \equiv M^*_{ij}y_j(a) + N^*_{ij}y_j(b), \quad t^*_i[z] \equiv z_j(a)P^*_{ji} + z_j(b)Q^*_{ji}$$

are such that for arbitrary values of $y_i(x)$, $z_i(x)$ at $x = a$ and $x = b$ we have (Bliss [2], p. 565)

$$t^*_i[z]s_i[y] + t_i[z]s^*_i[y] = z_i(x)y_i(x)\,|_a^b. \tag{8.3}$$

Now let $B_{ij}(x)$ be arbitrary continuous functions such that the differential system

$$y'_i - B_{ij}(x)y_j = 0, \quad s_i[y] = 0 \qquad (i, j = 1, \cdots, m) \tag{8.4}$$

is incompatible. Such functions may be chosen in infinitely many ways. Let $y_i = Y_{ij}(x)$ $(j = 1, \cdots, m)$ be a fundamental set of solutions of the differential equations of (8.4), and set $\| D_{ij} \| = \| s_i(Y_j) \|$. We shall denote by $\| G_{ij}(x, t) \|$ the Green's matrix for the incompatible system (8.4).

If $f_i(x)$ are arbitrary continuous functions and h_i are given constants, then every solution $y_i = y_i(x)$ $(i = 1, \cdots, m)$ of the non-homogeneous system

$$L_i[y] = f_i(x), \quad s_i[y] = h_i \qquad (i = 1, \cdots, m) \tag{8.5}$$

is also a solution of the integral system

$$y_i(x) = \int_a^b K_{ij}(x, t)y_j(t)\,dt + F_i(x), \tag{8.6}$$

where

$$K_{ij}(x, t) = G_{ik}(x, t)[A_{kj}(t) - B_{kj}(t)] + \int_a^b G_{ik}(x, s)C_{kj}(s, t)\,ds,$$

$$F_i(x) = \int_a^b G_{ij}(x, t)f_j(t)\,dt + Y_{ik}(x)D^{-1}_{kj}h_j,$$

and conversely. Suppose that the system (8.1) has only the identically vanishing solution. Then the homogeneous integral system

$$y_i(x) \equiv \int_a^b K_{ij}(x, t)y_j(t)\,dt \tag{8.7}$$

has only the trivial solution $y_i(x) \equiv 0$, and there exists a resolvent matrix $\| R_{ij}(x, t) \|$ for the system (8.6). It may be shown that if (8.1) is incompatible, then for arbitrary continuous functions $f_i(x)$ and constants h_i the system (8.5) has a unique solution given by

$$y_i(x) = \int_a^b \mathcal{G}_{ij}(x, t)f_j(t)\,dt + c_{ij}(x)h_j, \tag{8.8}$$

where

3

$$
(8.9)\qquad
\begin{aligned}
\mathcal{G}_{ij}(x,t) &= G_{ij}(x,t) + \int_a^b R_{ik}(x,s)G_{kj}(s,t)\,ds,\\
c_{ij}(x) &= [Y_{il}(x) + \int_a^b R_{ik}(x,s)Y_{kl}(s)\,ds]D^{-1}{}_{lj}.
\end{aligned}
$$

Corresponding to the notation of Tamarkin [21] and Jonah [11], the matrix $\| \mathcal{G}_{ij}(x,t) \|$ will be called a Green's matrix for the integro-differential boundary problem (8.1). It follows readily that for an incompatible system (8.1) the Green's matrix is unique, and hence independent of the particular choice of the matrix $\| B_{ij} \|$.

Using the properties of the ordinary Green's matrix for the differential system (8.4), together with results of the Fredholm integral equation theory, one may prove the following results.

THEOREM 8.1. *The number of linearly independent solutions of the homogeneous system* (8.1) *is the same as the number of such solutions of the adjoint system* (8.2).

THEOREM 8.2. *If the integro-differential systems* (8.1) *and* (8.2) *are incompatible, and* $\| \mathcal{G}_{ij}(x,t) \|$ *and* $\| \mathcal{H}_{ij}(x,t) \|$ *are the respective Green's matrices, then* $\mathcal{G}_{ij}(x,t) = -\mathcal{H}_{ji}(t,x)$.

THEOREM 8.3. *If the system* (8.1) *has exactly* r *linearly independent solutions* $y_i = y_{i\nu}(x)$ $(\nu = 1, \cdots, r)$, *then the non-homogeneous system* (8.5) *has a solution if and only if the condition*

$$
(8.10)\qquad \int_a^b z_i(x)f_i(x)\,dx = h_i t^*{}_i[z]
$$

is satisfied by every solution $z_i(x)$ *of the adjoint system* (8.2); *moreover, the most general solution of* (8.5) *is of the form*

$$
y_i = y^*{}_i(x) + y_{i\nu}(x)c_\nu,
$$

where $y^*{}_i(x)$ *is a particular solution and the* c_ν*'s are arbitrary constants.*

If the system (8.1) is compatible one may define a generalized Green's matrix for this system, using methods similar to those employed by the author [17] in considering compatible differential systems. Since, however, no use is made of generalized Green's matrices in the consideration of the problem defined in Section 2, the explicit form of the generalized Green's matrix will not be given.

A boundary problem (8.1) will be said to be self-adjoint if there exists a non-singular matrix $\| T_{ij}(x) \|$ whose elements are of class C^1 on $a \leq x \leq b$, and satisfying the conditions

$$(8.11)\quad \begin{aligned} &T_{ik}A_{kj} + A_{ki}T_{kj} + T'_{ij} \equiv 0, \quad T_{ik}(x)C_{kj}(x,t) + T_{kj}(t)C_{ki}(t,x) \equiv 0 \\ &M_{ik}T^{-1}_{kl}(a)M_{jl} - N_{ik}T^{-1}_{kl}(b)N_{jx} = 0 \qquad (i, j = 1, \cdots, m). \end{aligned}$$

This definition of self-adjointness corresponds to that given by Bliss ([2], p. 569) for a differential boundary problem. If a matrix $\| T_{ij}(x) \|$ satisfies the above conditions it follows readily that for every solution $y_i(x)$ of (8.1) there is a solution $z_i(x)$ of (8.2) given by $z_i(x) = T_{ij}(x)y_j(x)$, and conversely. In a manner similar to that used by Bliss [2] to prove the corresponding result for a differential boundary problem, one may establish the following theorem.

THEOREM 8.4. *Suppose the problem* (8.1) *is incompatible and is self-adjoint with respect to the matrix* $\| T_{ij}(x) \|$. *Then the functions of the Green's matrix* $\| \mathfrak{G}_{ij}(x,t) \|$ *satisfy the relations*

$$(8.12)\qquad T_{ik}(x)\mathfrak{G}_{kj}(x,t) + T_{kj}(t)\mathfrak{G}_{ki}(t,x) \equiv 0.$$

As indicated in Section 3, the integro-differential system (2.4′), (2.5′) is self-adjoint with respect to the constant matrix (3.6).

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A PARTIAL DIFFERENTIAL EQUATION ASSOCIATED WITH POISSON'S WORK ON THE THEORY OF SOUND.*

By H. Bateman.

Introduction. In his famous memoir of 1808 on the theory of sound, in which he discussed the theory of sound waves of finite amplitude, Poisson [1] also made some important advances in the theory of sound waves of small amplitude. In particular he attacked the problem of solving the equation of wave-propagation in three dimensions by using the mean value of a function over a sphere; a method which eventually led to a general solution and proved very fruitful in potential theory. This method has been supplemented by the consideration of mean values around circles on a sphere.[2]

To determine the velocity components of the individual particles of air Poisson tried to solve the wave equation by means of an infinite series of powers of the inverse distance in which the n-th coefficient is an integral of order n. The integrands of the various integrals are connected by a recurrence relation of the first order involving partial derivatives of the first two orders. Poisson found that these integrands could be obtained from a generating function which in turn satisfies a certain partial differential equation, namely, the one designated as equation (1) below. Incidentally, this differential equation also occurs in the above mentioned theory of mean values around circles on a sphere.

Here it is pointed out that the same partial differential equation may be derived from the wave equation by a simple transformation suggested by Poisson's work. The general problem of obtaining a wave function from a solution of (1) is considered. It is found that $(1/r)U(w, \mu, \phi)$, where $\mu = \cos\theta$ and r, θ, ϕ are polar coördinates, is a wave function provided w is defined by equation (6).

Equation (1) has particular solutions represented by products of Legendre functions. By comparing these solutions with another solution of the wave equation expansions involving Legendre functions are suggested. The determi-

* Received December 6, 1937.

[1] S. D. Poisson, *Bull. Soc. Philon.*, vol. 1 (1807), p. 19; *Journal de l'École Polytechnique*, t. 7, Cah. 14 (1808), pp. 319-392.

[2] H. Bateman, *Proceedings of the National Academy of Sciences*, vol. 16 (1930), pp. 205-211; *Annals of Mathematics* (2), vol. 31 (1930), pp. 158-162.

nation of the coefficients in these expansions requires the evaluation of a class of integrals discussed in an accompanying paper.

Relation to the wave equation. The differential equation which forms the subject of this paper is

$$(1)\qquad \frac{\partial}{\partial w}\left[(1-w^2)\frac{\partial U}{\partial w}\right]=\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial U}{\partial \mu}+\frac{1}{1-\mu^2}\frac{\partial^2 U}{\partial \phi^2}.$$

Poisson's work indicates that the wave equation

$$(2)\qquad \frac{\partial^2 W}{\partial t^2}=\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}$$

has particular solutions of the type

$$(3)\qquad W=\frac{1}{r}U(w,\mu,\phi)$$

where U is a solution of (1), $\mu=\cos\theta$ and r, θ, ϕ are polar coördinates, and w is defined by

$$(4)\qquad w=\frac{t}{r}.$$

This may be verified by transforming the wave equation by the substitution $z=\mu r$, $r^2=x^2+y^2+z^2$, $y=x\tan\phi$ after which it takes the form

$$(5)\quad \frac{\partial^2(rW)}{\partial t^2}=\frac{\partial^2(rW)}{\partial r^2}+\frac{1}{r^2}\left[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}(rW)\right]+\frac{1}{r^2(1-\mu^2)}\frac{\partial^2}{\partial\phi^2}(rW).$$

Direct substitution of (3) and (4) in (5) then results in equation (1).

When we consider the general problem of finding a function $w=w(r,t)$ such that $W=(1/r)U(w,\mu,\phi)$ satisfies equation (5), and hence is a wave function, we are led to the two equations

$$\frac{\partial^2 w}{\partial t^2}=\frac{\partial^2 w}{\partial r^2}-\frac{2w}{r^2},\qquad \left(\frac{\partial w}{\partial t}\right)^2-\left(\frac{\partial w}{\partial r}\right)^2=\frac{1-w^2}{r^2}.$$

If we seek a solution of the type $w=\sum_{n=0}^{\infty}A_n(r)t^n$ we find that

$$(6)\qquad rw=C(r^2-t^2)+D+t(1-4CD)^{\frac{1}{2}}$$

where C and D are arbitrary constants. When $C=D=0$, Poisson's expression $w=t/r$ is obtained, while if $C=1/2a$, $D=a/2$ the value [3]

[3] H. Bateman, *loc. cit.*

$$w = \frac{r^2 - t^2 + a^2}{2ar} \tag{7}$$

is obtained.

An expression suggested by a solution of equation (1). The differential equation (1) has solutions of the type

$$\begin{aligned} U &= P_n(w) P_n{}^m(\mu) e^{im\phi} \\ U &= P_n(w) Q_n{}^m(\mu) e^{im\phi} \end{aligned} \tag{8}$$

where m and n are arbitrary constants. We shall take them to be non-negative integers. When we replace μ by $\tanh v$, the corresponding solutions of the wave equation are

$$\begin{aligned} W &= \frac{1}{r} P_n(w) P_n{}^m(\tanh v) e^{im\phi} \\ W &= \frac{1}{r} P_n(w) Q_n{}^m(\tanh v) e^{im\phi}. \end{aligned} \tag{9}$$

Another solution may be obtained directly from the wave equation by assuming that W is independent of z. Upon setting $Z = t$, $X = ix$, $Y = iy$ the wave equation reduces to Laplace's equation which is known to be satisfied by

$$W = \frac{1}{R} F\left(\frac{Z + R}{X + iY}\right), \qquad R^2 = X^2 + Y^2 + Z^2.$$

The values of $X + iY$ and R^2 expressed in terms of r, w, v and ϕ, where we now take $w = t/r$, are

$$\begin{aligned} X + iY &= ir \operatorname{sech} v e^{i\phi} \\ R^2 &= t^2 - x^2 - y^2 = r^2(w^2 - \operatorname{sech}^2 v) \end{aligned}$$

and hence the expression above for W assumes the form

$$W = \frac{1}{r\sqrt{w^2 - \operatorname{sech}^2 v}} f\left[e^{-i\phi}\{w\, ch\, v + (w^2 ch^2 v - 1)^{\frac{1}{2}}\}\right]$$

where $F(-is) = f(s)$. The argument of the function f may be simplified by writing $ch\, u = w\, ch\, v$. Then

$$W = \frac{1}{r\sqrt{w^2 - \operatorname{sech}^2 v}} f(e^{-i\phi + u}).$$

The conjugate complex of this expression is also a solution of the wave equation.

$$W = \frac{1}{r\sqrt{w^2 - \text{sech}^2 v}} f_1(e^{i\phi+u}).$$

Taking $f_1(s) = s^m/2$ and $f(s) = s^{-m}/2$ and adding shows that

$$W = \frac{e^{im\phi}}{r\sqrt{w^2 - \text{sech}^2 v}} ch\, mu = \frac{e^{im\phi} T_m(w\, ch\, v)}{r\sqrt{w^2 - \text{sech}^2 v}} \tag{10}$$

is also a solution. $T_m(z)$ is Tchebycheff's polynomial defined as

$$T_m(z) = ch(m\, ch^{-1} z).$$

The solution (10) may be expanded in terms of the simple solutions (9). For example, when m is even, say $m = 2k$,

$$\sum_{n=0}^{\infty} (4n+1) P_{2n}(w) Q_{2n}^{2k}(\tanh v) / Q_{2n}^{2k+1}(0) = -\frac{T_{2k}(w\, ch\, v)}{r\sqrt{w^2 - \text{sech}^2 v}} \text{ or } 0,$$
$$-1 < w < 1$$

the first or second value on the right being taken accordingly as $w^2 - \text{sech}^2 v$ is positive or negative. The determination of the cofficients in this series requires the evaluation of the integral

$$\int_0^v du\, P_{2n}\left(\frac{ch\, u}{ch\, v}\right) ch(2ku)\, du$$

which is discussed in an accompanying paper.

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INTEGRALS INVOLVING LEGENDRE FUNCTIONS.*

By H. BATEMAN and S. O. RICE.

The preceding paper by one of us investigates the partial differential equation

$$\frac{\partial}{\partial w}\left[(1-w^2)\frac{\partial U}{\partial w}\right]=\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial U}{\partial \mu}\right]+\frac{1}{1-\mu^2}\frac{\partial^2 U}{\partial \phi^2}$$

which occurs in Poisson's work on the theory of sound. The integrals derived in the present paper are closely associated with this differential equation and give some of the ground work for a complete discussion of it. They have been evolved gradually from an initial clue suggested by some work of Beltrami on symmetrical potential functions, one author making one step and the other the next and so on.

Outline of results. The comparison of two types of integrals, due to Beltrami and Laplace respectively, for symmetrical potential functions suggests the existence of the relations stated in equations (4) and (5). These equations may then be verified directly. It is also possible to establish them in a modified form by complex integration.

These identities are suitable for dealing with Legendre functions, and when applied to an integral which follows from Hobson's addition theorem for $Q_n(z)$, they lead to the integrals given by equations (10) and (11). By analytic continuation of the parameters in these integrals still other integrals are derived. Equation (22) was first obtained by applying the recurrence relations for Legendre functions to the four integrals (14), (15), (18) and (19). However, this work is omitted, as here it is more convenient to regard (22) as a special case of equation (35).

Up to and including equation (22) the degree n and order m of every Legendre function are integers. A study of equation (22) with the idea of extending it to non-integer values of n and m led to theorems concerning contour-integrals having Legendre functions in the integrand. An application of one of these theorems leads to the desired extension given by equation (35). The theorem also suggests the result given by equation (36).

The paper ends with the derivation of an expansion for which (22) is

* Received December 6, 1937.

used to determine the coefficients. In the accompanying paper the existence of this expansion is suggested by solutions of the wave equation.

Potential functions having radial symmetry about an axis. If x, y, z are rectangular coördinates and $w = \sqrt{x^2 + y^2}$ then a potential function symmetrical about the z axis is obtained by integrating the elementary potential $1/R$ [1]

$$(1) \qquad V = \int_{-a}^{a} \frac{F(t)\,dt}{R}$$

where the constant a and the function $F(t)$ are arbitrary and

$$R^2 = w^2 + (z + it)^2.$$

Another symmetrical potential involving R may be obtained from Laplace's expression for a potential which assumes the value $F(z)$ along the Z axis,

$$V = \frac{1}{\pi} \int_0^{\pi} f(z + iw \cos k)\,dk.$$

Changing the variable leads to

$$(2) \qquad V = \frac{1}{\pi} \int_{iz-w}^{iz+w} \frac{f(-it)\,dt}{R}.$$

The similarity of the integrands suggests the existence of a relationship between the functions $F(t)$ and $f(-it)$. Consideration of the potential of a circular ring of radius a indicated that this relation is

$$(3) \qquad f(z) = \int_{-a}^{+a} \frac{F(t)\,dt}{z + it}.$$

That is, when (3) is satisfied, we expect the equality

$$(4) \qquad \int_{-a}^{a} dt \frac{F(t)}{R} = \frac{1}{\pi} \int_{iz-w}^{iz+w} \frac{f(-it)}{R}\,dt.$$

This may be verified by substituting the expression for $f(z)$ given by (3) in the right member and integrating after inverting the order of integration.

If instead of expressions (1) and (2) we start with

[1] This expression is closely associated with Beltrami's integrals for a symmetrical potential.

Rendiconti Lombardo (2), vol. 11 (1878), pp. 668-680 = *Opere*, vol. 3, pp. 115-128.

Bologna Mem. (4), vol. 2 (1880), pp. 461-505 = *Opere*, vol. 3, pp. 349-382.

Bologna Mem. (4), vol. 4 (1882), pp. 211-216 = *Opere*, vol. 4, pp. 45-76.

$$W = \int_{-a}^{+a} F(t)(z+it)\,dt/R$$

$$W = \frac{iw}{\pi}\int_{0}^{\pi} f(z+iw\cos k)\cos k\,dk,$$

the first of which is related to Beltrami's integrals, we obtain in place of (4)

$$(5)\quad \int_{-a}^{a} \frac{F(t)(z+it)\,dt}{R} - \int_{-a}^{a} F(t)\,dt = \frac{1}{\pi}\int_{iz-w}^{iz+w} \frac{(z+it)f(-it)\,dt}{R}$$

which may be verified in the same way as (4).

It is also possible to establish these identities, in a modified form, by using Cauchy's theorem. The result may be stated as follows:

Let $g(s)$ be continuous on the path D in the s-plane and let

$$(6)\qquad h(t) = \frac{1}{2\pi i}\int_{D} \frac{g(s)\,ds}{s-t}$$

then, if $S(t) = (t-t_1)(t-t_2)$, we have the relation

$$(7)\qquad \frac{1}{\pi}\int_{t_1}^{t_2} [-S(t)]^{-\frac{1}{2}}\,h(t)\,dt = \frac{1}{2\pi i}\int_{D} [S(s)]^{-\frac{1}{2}}\,g(s)\,ds$$

provided D does not cross the line joining t_1 and t_2. The arguments are chosen so that

$$\arg(t-t_1) = \arg(t_2-t) = \theta,$$

where θ is the angle which the vector from t_1 to t_2 makes with the real axis, and

$$\theta \leq \arg(s-t_1) < 2\pi+\theta, \qquad 0 \leq \arg(s-t_2) < 2\pi.$$

Under these conditions there is also an analogue of (5), namely

$$(8)\qquad \frac{1}{\pi}\int_{t_1}^{t_2} [-S(t)]^{-\frac{1}{2}}\,h(t)\,t\,dt$$
$$= -\frac{1}{2\pi i}\int_{D} g(s)\,ds + \frac{1}{2\pi i}\int_{D} [S(s)]^{-\frac{1}{2}}\,g(s)\,s\,ds.$$

The first step in the proof of (7) is to replace $\frac{1}{\pi}\int_{t_1}^{t_2}(\ )\,dt$ by $\frac{1}{2\pi i}\int_{C}(\ )\,dt$ where C is a contour enclosing the points t_1 and t_2 which does not cross D. Use of (6) and a change of order of integration, which may be justified, leads to the integral

$$\frac{1}{2\pi i}\int_{C} \frac{[S(t)]^{-\frac{1}{2}}\,dt}{s-t}$$

which may be evaluated by expanding C until it consists of an infinite circle plus a loop about $t = s$. Only the latter contributes to the value of the integral, which is seen to be $[S(s)]^{-\frac{1}{2}}$. The result stated is then obtained by letting C, on the other side of the equation, shrink to a dumbell shaped contour with very small loops around t_1 and t_2.

Equation (8) is proved in an analogous manner.

Application of the identities to integrals involving Legendre functions. The fact that Legendre's functions of the first and second kinds are related, for integer values of n, by

$$Q_n(t) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(s)\,ds}{t - s}$$

enables us to apply the identities to these functions. For upon setting $h(t) = Q_n(t)$, $g(s) = -\pi i P_n(s)$ and taking D to be the line joining -1 and $+1$ we see that equation (6) becomes the one just given. To obtain an integral of the type used in the identities we note from Hobson's addition theorem for $Q_n(z)$ [2] it follows that

$$Q_n(b)P_n(c) = \frac{1}{\pi}\int_0^{\pi} Q_n(bc - (b^2-1)^{\frac{1}{2}}(c^2-1)^{\frac{1}{2}}\cos a)\,da$$

where n is an integer and $b > c > 1$. Changing the variable of integration transforms the integral into

$$Q_n(b)P_n(c) = \frac{1}{\pi}\int_{t_1}^{t_2} \frac{Q_n(t)\,dt}{\sqrt{(t_2 - t)(t - t_1)}} \tag{9}$$

where

$$t_2 = bc + \sqrt{(b^2-1)(c^2-1)}, \qquad t_1 = bc - \sqrt{(b^2-1)(c^2-1)}.$$

Since $t_2 > t_1 > 1$, the path D joining -1 and $+1$ does not cross the line joining t_2 and t_1. Also $P_n(s)$ is continuous on D and we may apply equation (7) to transform the integral in equation (9). Thus noting that $\theta = 0$ and $\arg(s - t_1) = \arg(s - t_2) = \pi$ we obtain the known result

$$\begin{aligned} 2Q_n(b)P_n(c) &= \int_{-1}^{+1} \frac{P_n(s)\,ds}{\sqrt{(t_2 - s)(t_1 - s)}} \\ &= \int_{-1}^{+1} \frac{P_n(s)\,ds}{\sqrt{b^2 + c^2 + s^2 - 2bcs - 1}}. \end{aligned} \tag{10}$$

[2] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics,* Cambridge (1931), p. 378. In the following work this text will be denoted by Hobson (1).

In the same way we obtain from (8) and

$$Q_n^1(b)P_n^{-1}(c) = \frac{1}{\pi}\int_0^{\pi} Q_n(bc - \sqrt{(b^2-1)(c^2-1)}\cos a)\cos a\, da$$

the equation

$$(11)\quad 2Q_n^1(b)P_n^{-1}(c) = \int_{-1}^{+1} \frac{(s-bc)P_n(s)ds}{\sqrt{(b^2-1)(c^2-1)}\sqrt{b^2+c^2+s^2-2bcs-1}}$$

which holds when n is a positive integer.

Analytic continuation applied to the integrals in equations (10) and (11). When the integrals in equations (10) and (11) are viewed as analytic functions of the parameters b and c, some interesting integrals may be obtained by continuing b and c from their original values, which were greater than unity, to values lying between -1 and $+1$. For the time being we shall denote these new values by b_1 and c_1 respectively and assume that $b_1 > c_1$.

In the expressions for t_1 and t_2 we first let c decrease to c_1 along a path which lies on the real axis except for an indentation upward at unity. Then the corresponding values of t_1 and t_2 are

$$t_2 = bc_1 + i\sqrt{(b^2-1)(1-c_1^2)}, \qquad t_1 = bc_1 - i\sqrt{(b^2-1)(1-c_1^2)}$$

When we let b decrease to b_1 along a similar path we obtain

$$t_2 = b_1c_1 - \sqrt{(1-b_1^2)(1-c_1^2)}, \qquad t_1 = b_1c_1 + \sqrt{(1-b_1^2)(1-c_1^2)}$$

where now both t_1 and t_2 lie between -1 and $+1$ and $t_1 > t_2$. Incidentally, the reason for keeping $b_1 > c_1$ becomes apparent upon letting $b_1 = c_1$ for then $t_2 = -1$ and $t_1 = +1$.

We must take care that in this process neither t_2 nor t_1 crosses the path of integration D of s in (10) and (11). A more detailed examination of the paths followed by t_2 and t_1 shows that t_2 must lie above and t_1 below D. For this reason D is indented downward at $s = t_2$ and upward at $s = t_1$.

Since the path followed by b in decreasing to b_1 was indented so as to pass above the point $+1$, b_1 will lie above the cut in the z plane joining -1 to $+1$ associated with $Q_n(z)$. Thus, following the usual custom, we write $Q_n(b_1 + i0)$.

Splitting the integral (10) up into its real and imaginary parts we obtain, upon dropping the subscript one,

$$(12)\quad 2Q_n(b+i0)P_n(c) = \left\{\int_{-1}^{t_2} + \int_{t_1}^{1}\right\} \frac{ds\,P_n(S)}{\sqrt{b^2+c^2+s^2-2bcs-1}} - i\int_{t_2}^{t_1} \frac{ds\,P_n(s)}{\sqrt{1+2bcs-b^2-c^2-s^2}}$$

where now $-1 < c < b < 1$. The function $Q_n(b + i0)$ may be split into its real and imaginary parts by utilizing the definition of $Q_n(z)$ when $-1 < z < 1$;

$$Q_n(z) = Q_n(z + i0) + \frac{\pi i}{2} P_n(z).$$

Equating imaginary parts gives, after a transformation inverse to that used in obtaining equation (9), the result

$$P_n(b)P_n(c) = \frac{1}{\pi}\int_0^\pi P_n[bc - (1-b^2)^{\frac{1}{2}}(1-c^2)^{\frac{1}{2}}\cos a]\, da.$$

This equation may be readily obtained from the addition theorem for Legendre polynomials.

Setting $b = \tanh u$, $c = \tanh v$ and equating imaginary parts gives

$$(13) \qquad \begin{aligned} &2Q_n(\tanh u)P_n(\tanh v) \\ &\quad = \int_0^{u+v} P_n\left(\tanh u \tanh v - \frac{ch\, t}{ch\, u\, ch\, v}\right) dt \\ &\quad + \int_0^{u-v} P_n\left(\tanh u \tanh v + \frac{ch\, t}{ch\, u\, ch\, v}\right) dt \end{aligned}$$

where it is assumed that $u > v > 0$ in the derivation. Since both sides are analytic functions of v in the neighborhood of the real axis this restriction may be removed by analytic continuation.

The following are special cases of equation (13)

$$(14) \qquad Q_{2n+1}(0)P_{2n+1}(\tanh v) = -\int_0^v P_{2n+1}\left(\frac{ch\, t}{ch\, v}\right) dt$$

$$(15) \qquad Q_{2n}(\tanh u)P_{2n}(0) = \int_0^u P_{2n}\left(\frac{ch\, t}{ch\, u}\right) dt$$

$$Q_n(\tanh u)P_n(\tanh u) = \int_0^u P_n\left(1 - 2\frac{ch^2\, s}{ch^2\, u}\right) ds.$$

In the same way equation (11) leads to the analogue of (13)

$$(16) \qquad \begin{aligned} &2 \text{ Real Part of } Q_n^1(\tanh u + i0)P_n^{-1}(\tanh v + i0) \\ &\quad = \int_0^{u-v} P_n\left(\tanh u \tanh v - \frac{ch\, t}{ch\, u\, ch\, v}\right) ch\, t\, dt \\ &\quad - \int_0^{u-v} P_n\left(\tanh u \tanh v + \frac{ch\, t}{ch\, u\, ch\, v}\right) ch\, t\, dt. \end{aligned}$$

By using the following results [3] which hold for general values of m and n

[3] The first two are given by E. W. Hobson, *Philosophical Transactions*, vol. 187 (1896), pp. 443-531, equations (19) and (29). The third one is given Hobson (1), p. 229.

$$(17)\quad P_n^{-m}(x+iy)=\frac{\Pi(n-m)}{\Pi(n+m)}\left\{P_n^m(x+iy)-\frac{2}{\pi}e^{-m\pi i}\sin m\pi\, Q_n^m(x+iy)\right\}$$
$$P_n^m(x+i0)=e^{-m\pi i/2}P_n^m(x)$$
$$Q_n^m(x+i0)=e^{3m\pi i/2}\left\{Q_n^m(x)-\frac{i\pi}{2}P_n^m(x)\right\}$$

where x and y are real numbers, the left-hand side of (16) may be written as

$$-\frac{2}{n(n+1)}Q_n^1(\tanh u)P_n^1(\tanh v).$$

Special cases of (16) are

$$(18)\quad -\frac{1}{2n(2n+1)}Q_{2n}^1(0)P_{2n}^1(\tanh v)=\int_0^v P_{2n}\left(\frac{ch\,t}{ch\,v}\right)ch\,t\,dt$$

$$(19)\quad \frac{1}{(2n+1)(2n+2)}Q^1{}_{2n+1}(\tanh u)P^1{}_{2n+1}(0)=\int_0^u P_{2n+1}\left(\frac{ch\,t}{ch\,u}\right)ch\,t\,dt$$

$$(20)\quad -\frac{1}{n(n+1)}Q_n^1(\tanh u)P_n^1(\tanh u)=\int_0^u P_n\left(1-\frac{2ch^2s}{ch^2u}\right)ch\,2s\,ds.$$

By applying the recurrence relation for the Legendre functions to equations (14), (15), (18), (19) and using the expressions [4]

$$(21)\quad \begin{aligned} P_n^m(0)&=2^m\cos\frac{n+m}{2}\pi\,\frac{\Pi\left(\frac{n+m-1}{2}\right)}{\Pi\left(\frac{n-m}{2}\right)\Pi(-\frac{1}{2})}\\ Q_n^m(0)&=-2^{m-1}\sin\frac{n+m}{2}\pi\,\frac{\Pi\left(\frac{n+m-1}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m}{2}\right)}\end{aligned}$$

the following relation may be established

$$(22)\quad \int_0^v P_n\left(\frac{ch\,u}{ch\,v}\right)ch\,mu\,du=\begin{cases}-Q_n^m(\tanh v)/Q_n^{m+1}(0), & m+n \text{ even}\\ -P_n^m(\tanh v)/P_n^{m+1}(0), & m+n \text{ odd.}\end{cases}$$

In this equation m and n are assumed to be positive integers.

Further integrals involving Legendre functions. Equation (22) suggests the problem of determining the value of the integral when m and n are not integers. Since none of the foregoing work may be applied when m and n are not integers we must search for a different method of attack. A study of (22) suggests the two theorems

[4] Hobson (1), p. 232.

1. If $\phi(x)$ be a Legendre function of order n, the integral

$$(23) \qquad I = \int_P \frac{\phi(au)\exp(m\, ch^{-1}u)\, du}{\sqrt{u^2-1}}$$

may be expressed in terms of the associated Legendre functions:

$$(24) \qquad I = AP_n{}^m(b) + BQ_n{}^m(b)$$

where $b = \sqrt{1-a^2}$ and A and B are independent of a, provided the expression

$$(25) \qquad F(u) = [a\sqrt{u^2-1}\,\phi'(au) - m\phi(au)]\exp(m\, ch^{-1}u)$$

has the same value at the initial and final points, both assumed independent of a, of the path P. The prime denotes differentiation: $\phi'(t) = \frac{d\phi(t)}{dt}$.

2. The same is true of the integral

$$(26) \qquad J = \int_P \frac{\phi(t)\exp\left(m\, ch^{-1}\frac{t}{a}\right) dt}{\sqrt{t^2-a^2}}$$

provided the expression

$$(27) \qquad G(t) = (t^2-1)\left[\frac{\phi'(t)}{z} + \frac{t\phi(t)}{z^3} - \frac{m\phi(t)}{z^2}\right]\exp\left(m\, ch^{-1}\frac{t}{a}\right)$$

where $z = \sqrt{t^2-a^2}$, assumes the same value at the initial and final points, which are here also taken to be independent of a, of the path P.

Although these results were found more or less by experiment, they may be verified by straightforward differentiation. Thus it may be shown that when I and J are placed in the differential equation for the associated Legendre functions we obtain

$$\frac{d}{db}(1-b^2)\frac{dI}{db} + \left\{n(n+1) - \frac{m^2}{1-b^2}\right\}I = \frac{1}{a^2}\int_P du\,\frac{d}{du}F(u)$$
$$\frac{d}{db}(1-b^2)\frac{dJ}{db} + \left\{n(n+1) - \frac{m^2}{1-b^2}\right\}J = \int_P dt\,\frac{d}{dt}G(t)$$

and if $F(u)$ and $G(t)$ assume the same values at the initial and final points of their respective paths the integrals on the right vanish.

Since m occurs in the differential equation as m^2 we may change the sign of m in I and J. Addition and subtraction will then give theorems concerning integrals having hyperbolic cosines and sines in place of the exponential functions in I and J. In particular we obtain the result that

$$K = \int_P \phi(t) ch \left(m\, ch^{-1} \frac{t}{a}\right) \frac{dt}{\sqrt{t^2 - a^2}} \tag{28}$$

may be expressed in the form (24) provided

$$H(t) = (t^2 - 1) \left[\left(\frac{\phi'}{z} + \frac{t\phi}{z^3}\right) ch \left(m\, ch^{-1} \frac{t}{a}\right) - \frac{m\phi}{z^2}\, sh \left(m\, ch^{-1} \frac{t}{a}\right)\right] \tag{29}$$

assumes the same value at the initial and final points of P.

In order to generalize equation (22) we consider the expression (28) where $\phi(t) = P_n(t)$ and P is a path which starts at $t = 1$, passes around $t = a$ in the positive direction, and returns to the point $t = 1$. For the sake of convenience we assume that a is real, $0 < a < 1$, and $\arg(t - a) = \arg(t + a) = 0$ at the starting point. The expression (29) vanishes at both ends of P, because of the factor $t^2 - 1$, and hence K may be expressed in the form (24). Instead of K it is more suitable to deal with L defined by

$$L = \int_a^1 P_n(t) ch \left(m\, ch^{-1} \frac{t}{a}\right) \frac{dt}{\sqrt{t^2 - a^2}} \tag{30}$$

where the square root is supposed to be positive.

By shrinking P to a small loop about $t = a$ plus two straight lines joining the circle to $t = 1$ we see that

$$L = -\frac{1}{2} K$$

and thus L may also be expressed as

$$L = A P_n^m(\sqrt{1 - a^2}) + B Q_n^m(\sqrt{1 - a^2}). \tag{31}$$

As $a \to 1$, $L \to 0$ since $ch^{-1}\left(\frac{1}{a}\right) \to 0$ and $P_n(1) = 1$. Therefore

$$0 = A P_n^m(0) + B Q_n^m(0) \tag{32}$$

is one relation between A and B. To obtain a second relation we multiply the numerator and denominator of (30) by t and use

$$\frac{t\, dt}{\sqrt{t^2 - a^2}} = d(\sqrt{t^2 - a^2})$$

to integrate by parts. This leads to

$$\frac{dL}{da} \to -\frac{1}{\sqrt{1 - a^2}} \text{ as } a \to 1.$$

When this is placed in (31) after that equation has been differentiated, cancellation of the common factor $\frac{1}{\sqrt{1-a^2}}$ gives, upon letting $a \to 1$,

$$(33) \qquad 1 = \left[A \frac{d}{dx} P_n^m(x) + B \frac{d}{dx} Q_n^m(x) \right]_{x=0}.$$

Equations (32) and (33) may now be solved for A and B. If C denotes the constant in

$$P_n^m(x) \frac{d}{dx} Q_n^m(x) - Q_n^m(x) \frac{d}{dx} P_n^m(x) = \frac{C}{1-x^2}.$$

The value of L is

$$(34) \qquad L = \frac{1}{C} \left[P_n^m(0) Q_n^m(\sqrt{1-a^2}) - Q_n^m(0) P_n^m(\sqrt{1-a^2}) \right].$$

The values of $P_n^m(0)$ and $Q_n^m(0)$ are given by (21), while [5]

$$C = 2^{2m} \frac{\Pi\left(\frac{n+m-1}{2}\right) \Pi\left(\frac{n+m}{2}\right)}{\Pi\left(\frac{n-m-1}{2}\right) \Pi\left(\frac{n-m}{2}\right)}$$

When Hobson's definitions of the associated Legendre functions for $-1 < x < 1$ are used, substitution of the values for C, $P_n^m(0)$, $Q_n^m(0)$ in (34) and the change of variable $t = a \, ch \, u$, $ch \, v = 1/a$ in (30) gives for the final result

$$(35) \qquad \int_0^v P_n\left(\frac{ch \, u}{ch \, v}\right) ch \, m \, u \, du$$
$$= \frac{\sqrt{\pi}\, \Pi\left(\frac{n-m-1}{2}\right)}{2^m\, \Pi\left(\frac{n+m}{2}\right)} \left[\frac{\cos(n+m)\frac{\pi}{2}}{\pi} Q_n^m(\tanh v) + \frac{\sin(n+m)\frac{\pi}{2}}{2} P_n^m(\tanh v) \right]$$

which is a generalization of (22).

In a somewhat similar manner it may be shown that

$$(36) \qquad \int_0^\infty Q_n(a \, ch \, u) \, ch \, m \, u \, du = Q_n^{-m}(0 + i0) Q_n^m(-i\sqrt{a^2-1})$$
$$= \frac{\sqrt{\pi}\, \Pi\left(\frac{n-m-1}{2}\right)}{2^{m+1}\, \Pi\left(\frac{n+m}{2}\right)} e^{-i\pi[m+(n+1)/2]} Q_n^m(-i\sqrt{a^2-1})$$

[5] Hobson (1), p. 232.

where $R(n+1-m)>0$, $R(n+m+1)>0$, which imply $R(n)>-1$; and $\arg(\sqrt{a^2-1})=0$ when $\arg a=0$ and $|a|>1$. Equation (36) is suggested by (28) when $\phi(t)$ is set equal to $Q_n(t)$ and P is taken to be a path starting and ending at $+\infty$ which comes in and encloses $t=a$. It may be readily verified when $|a|>1$ by replacing $Q_n(a\, ch\, u)$ by its series expansion in powers of $1/a\, ch\, u$ and integrating termwise.

Expansions suggested by equation (22). Since a function which satisfies Darboux's conditions [6] may be expanded in the Legendre series [7]

$$f(z)=\sum_{n=0}^{\infty} a_n P_n(z)(n+\tfrac{1}{2}) \tag{37}$$

where

$$a_n=\int_{-1}^{+1} f(t)P_n(t)\,dt, \tag{38}$$

and since (22) is of the form (38), it is natural to seek the corresponding function $f(z)$. By setting $t=chu/chv$, $a=\operatorname{sech} v$ we see that the integral in (22) is changed into the integral occurring in (30) and hence we are led to set

$$\begin{aligned} f(t) &= ch\left(m\, ch^{-1}\frac{t}{a}\right)/\sqrt{t^2-a^2}=T_m(t/a)/\sqrt{t^2-a^2}, \quad a<t<1\\ &=0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad ,\ -1<t<a \end{aligned}$$

where $T_m(z)$ is Tchebycheff's polynomial which is such that when $z=ch\,u$, $T_m(z)=ch\,mu$. When m is even, say equal to $2k$, the expansion written in full is

$$\left.\begin{array}{ll} \operatorname{sech} v<t<1, & \dfrac{T_{2k}(t\,ch\,v)}{\sqrt{t^2-\operatorname{sech}^2 v}}\\ -1<t<\operatorname{sech} v, & 0 \end{array}\right\}=\begin{array}{l} -\sum\limits_{n=0}^{\infty} P_{2n}(t)(2n+\tfrac{1}{2})Q_{2n}^{2k}(\tanh v)/Q_{2n}^{2k+1}(0)\\ -\sum\limits_{n=0}^{\infty} P_{2n+1}(t)(2n+\tfrac{3}{2})P_{2n+1}^{2k}(\tanh v)/P_{2n+1}^{2k+1}(0). \end{array} \tag{39}$$

The first series represents an even function of t whereas the second one represents an odd function. When t is negative the sum of the two functions is zero. Hence when t is positive the two functions are equal. Separating the even and odd portions of (39) gives

$$\begin{aligned} &\sum_{n=0}^{\infty}(4n+1)P_{2n}(t)Q_{2n}^{2k}(\tanh v)/Q_{2n}^{2k+1}(0)=-\frac{T_{2k}(t\,ch\,v)}{\sqrt{t^2-\operatorname{sech}^2 v}} \text{ or } 0\\ &\sum_{n=0}^{\infty}(4n+3)P_{2n+1}(t)P_{2n+1}^{2k}(\tanh v)/P_{2n+1}^{2k+1}(0)=-\frac{T_{2k}(t\,ch\,v)}{t\sqrt{1-t^{-2}\operatorname{sech}^2 v}} \text{ or } 0. \end{aligned} \tag{40}$$

[6] G. Darboux, *Journal de Mathématique* (3), vol. 4 (1878), pp. 5-56, 377-416.

[7] E. W. Hobson, *Proceedings of the London Mathematical Society* (2), vol. 7 (1909), pp. 24-39.

The first or second value on the right is to be taken accordingly as $t^2 - \operatorname{sech}^2 v$ is positive or negative. The positive value of the square root is to be used. By considering m to be odd the expansions

$$(41)\quad \begin{aligned} \sum_{n=0}^{\infty}(4n+1)P_{2n}(t)P_{2n}^{2k+1}(\tanh v)/P_{2n}^{2k+2}(0) &= -\frac{T_{2k+1}(t\, ch\, v)}{\sqrt{t^2-\operatorname{sech}^2 v}} \text{ or } 0 \\ \sum_{n=0}^{\infty}(4n+3)P_{2n+1}(t)Q_{2n+1}^{2k+1}(\tanh v)/Q_{2n+1}^{2k+2}(0) &= -\frac{T_{2k+1}(t\, ch\, v)}{t\sqrt{1-t^{-2}\operatorname{sech}^2 v}} \text{ or } 0 \end{aligned}$$

are obtained.

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A NOTE ON AN EXTENSION OF BERNSTEIN'S THEOREM.*

By W. H. McEwen.

In a recent paper [1] the author obtained an extension of Bernstein's theorem for arbitrary sums of characteristic solutions of a general n-th order linear differential system

$$(1)\qquad \begin{aligned} &L(u) \equiv u^{(n)} + P_2(x)u^{(n-2)} + \cdots + P_n(x)u + \lambda u = 0, \\ &W_j(u) = 0 \qquad\qquad (j = 1, 2, \cdots, n), \end{aligned}$$

in which the coefficients $P_j(x)$ are continuous with continuous derivatives of all orders on (a, b), the boundary conditions are normalized and regular on (a, b), and the complex characteristic values λ_k (arranged in order of increasing moduli) give rise to poles of the Green's function which are simple when k is large.

The sums in question have the form $S_N(x) = \sum_{i=1}^{N} a_i u_i(x)$, in which the a's are arbitrary and the u's are the characteristic solutions of (1) corresponding respectively to the first N characteristic values $\lambda_1, \lambda_2, \cdots, \lambda_N$. Assuming that $| S_N(x) | \leqq L$ on (a, b) it was found that $| S'_N(x) | \leqq QNL$ uniformly on any interior interval $a + \delta \leqq x \leqq b - \delta$, Q being a constant independent of N, and an example was cited to show that this is the best result that can be obtained in general. In particular cases, however (as for example in the Fourier case or the Sturm-Liouville case), the limit QNL can be applied to the whole interval (a, b). In the present paper we propose to investigate further the circumstances under which this extension to the whole interval can be made. The discussion will be based on results found in a paper by Stone [2] (referred to hereafter as (S)), and the author's paper [3] (referred to as (M)).

Let us define, over the range $a \leqq a' \leqq b' \leqq b$, the function $Q(a', b')$, $0 \leqq Q(a', b') \leqq +\infty$, as the "best" constant such that

$$| S'_N(x) | \leqq Q(a', b')NL$$

* Received September 28, 1937.

[1] W. H. McEwen, "An extension of Bernstein's theorem associated with general boundary value problems," *American Journal of Mathematics*, vol. 59 (1937), pp. 295-305.

[2] M. H. Stone, "A comparison of the series of Fourier and Birkhoff," *Transactions of the American Mathematical Society*, vol. 28 (1926), pp. 695-761.

[3] *Loc. cit.*

for $a' \leqq x \leqq b'$, where L is the maximum value of $|S_N|$ on $a \leqq x \leqq b$. It is readily seen that the function Q has the following properties:

(i) if $a \leqq a'' \leqq a' \leqq b' \leqq b'' \leqq b$, then $Q(a', b') \leqq Q(a'', b'')$;
(ii) $Q(a', b') \rightarrow Q(a'', b'')$ when $a' \rightarrow a''$ and $b' \rightarrow b''$;
(iii) if $a \leqq a' \leqq b' \leqq c' \leqq b$, then $Q(a', c') = \max [Q(a', b'), Q(b', c')]$;
(iv) if $a < a' \leqq b' < b$, then $Q(a', b') < +\infty$.

The first three are implied by the definition of Q as the "best" constant, whereas the fourth was established in the author's earlier paper. From these properties it is evident that any further interest in the function Q centers in its behaviour as $a' \rightarrow a$ or $b' \rightarrow b$.

We can assume, without loss in generality, that the interval of x is $0 \leqq x \leqq 1$, and the maximum value of $|S_N(x)|$ on $(0,1)$ is 1. The boundary conditions of (1), being normalized, can be written

$$W_j(u) \equiv \alpha_j u^{(k_j)}(0) + \beta_j u^{(k_j)}(1) + \sum_{i=0}^{k_j-1} (\alpha_{ij} u^{(i)}(0) + \beta_{ij} u^{(i)}(1)) = 0, \quad (j = 1, 2, \cdots, n),$$

in which the k_j's are positive integers such that $n - 1 \geqq k_1 \geqq k_2 \geqq \cdots \geqq k_n$ and no three k_j's are the same. Along with (1) let us consider a second system of the same type and of the same order n:

$$\bar{L}(u) \equiv u^{(n)} + \bar{P}_2(x) u^{(n-2)} + \cdots + \bar{P}_n(x) u + \lambda u = 0,$$

$$(2) \qquad \bar{W}_j(u) \equiv \bar{\alpha}_j u^{(\bar{k}_j)}(0) + \bar{\beta}_j u^{(\bar{k}_j)}(1) + \sum_{i=0}^{k_j-1} (\bar{\alpha}_{ij} u^{(i)}(0) + \bar{\beta}_{ij} u^{(i)}(1)) = 0, \quad (j = 1, 2, \cdots, n),$$

and define

Hypothesis A. Systems (1) and (2) are so related that $\alpha_j = \bar{\alpha}_j$, $\beta_j = \bar{\beta}_j$ $k_j = \bar{k}_j$, $(j = 1, 2, \cdots, n)$.

Let $G(x, y; \lambda)$ and $\bar{G}(x, y; \lambda)$ be the Green's functions respectively of systems (1) and (2), and let $\lambda = \rho^n$. The sum $S_N(x)$ may then be written as a contour integral

$$S_N(x) = \frac{1}{2\pi i} \int_0^1 S_N(y) \int_\Gamma n \rho^{n-1} G(x, y; \rho^n) d\rho dy,$$

where Γ is an arc of the circle $|\rho| = R$ in the complex ρ-plane and is defined by two adjacent sectors of the set of $2n$ equal sectors:

$$l\pi/n \leqq \arg \rho \leqq (l+1)\pi/n \qquad (l = 0, 1, 2, \cdots, 2n-1).$$

The radius of Γ, $R \sim \pi N$, and is adjusted so that the arc remains uniformly away from the poles of both G and $\bar{G}$ when R (or N) is large.[4] The function

[4] See (M), p. 297.

$$\bar{S}_N(x) = \frac{1}{2\pi i}\int_0^1 S_N(y) \int_\Gamma n\rho^{n-1}\bar{G}(x, y; \rho^n)\, d\rho\, dy$$

will then represent the partial sum of order $\bar{N}$ of the Birkhoff expansion of the function $S_N(x)$ associated with system (2). Moreover the order $\bar{N}$ will satisfy $|N - \bar{N}| \leqq K$ so that $0(\bar{N}) = 0(N)$.

We now observe two lemmas:

LEMMA 1. *Under hypothesis A,*

$$\int_\Gamma n\rho^{n-1}(G - \bar{G})\, d\rho = 0(1)$$

uniformly on $0 \leqq x, y \leqq 1$.

LEMMA 2. *Under hypothesis A,*

$$\int_\Gamma n\rho^{n-1}\left(\frac{\partial G}{\partial x} - \frac{\partial \bar{G}}{\partial x}\right) d\rho = 0(N)$$

uniformly on $0 \leqq x, y \leqq 1$.

These lemmas are extensions respectively of lemmas 2 and 3 of (M), and are obtained from the latter by imposing the additional restrictions contained in hypothesis A. A proof of Lemma 1 for the case of a system of odd order is given in (S, Theorem XV, pp. 729-730). An analogous argument will suffice for the case of even order. The nature of the argument involved will be brought out in our outline of the proof of Lemma 2, for the case of odd order, $n = 2\mu - 1$, which follows:

From (M, p. 302) we obtain the formula

$$(3) \qquad n\rho^{n-1}\left(\frac{\partial G}{\partial x} - \frac{\partial \bar{G}}{\partial x}\right) \equiv \left\{ (F^0{}_{11} - \bar{F}^0{}_{11}) - \sum_{j=1}^{\mu} e^{\rho\omega_j(x-y)} \frac{(m_j - \bar{m}_j)}{\rho}; \right.$$
$$\left. (F^1{}_{11} - \bar{F}^1{}_{11}) + \sum_{j=\mu+1}^{n} e^{\rho\omega_j(x-y)} \frac{(m_j - \bar{m}_j)}{\rho} \right\}$$
$$+ \sum_{j=1}^{\mu} (\rho\omega_j) e^{\rho\omega_j x}(E_j - \bar{E}_j) + \sum_{j=\mu+1}^{n} (\rho\omega_j) e^{\rho\omega_j(x-1)}(E_j - \bar{E}_j),$$

which holds for values of ρ on the arc γ^1 (γ^1 is one of the halves of γ, which in turn is one of the halves of Γ. See M, p. 299). The functions $(F^0{}_{11} - \bar{F}^0{}_{11})$, $(F^1{}_{11} - \bar{F}^1{}_{11})$, m_j, $\bar{m}_j$, E_j, $\bar{E}_j$ and all the exponentials are uniformly bounded on $0 \leqq x, y \leqq 1$ as $R \to \infty$ (see M, p. 302). Hence the expression $\{\ \} = 0(1)$. On the other hand the summations in the last line of (3) are in general $0(R) = 0(N)$. However, under hypothesis A these also become $0(1)$, for in that case we find that $E_j - \bar{E}_j = 0(1)/\rho$.

To prove this last statement we note that the summations in question arise from the evaluation of the expression

$$(4) \qquad \frac{\Delta_1^{(1)}}{[\theta_1]e^{\rho\omega_\mu}+[\theta_0]}-\frac{\bar{\Delta}_1^{(1)}}{[\bar{\theta}_1]e^{\rho\omega_\mu}+[\bar{\theta}_0]},$$

in which the denominators are uniformly bounded away from zero as $R \to \infty$, and

$$\Delta_1^{(1)} \equiv \begin{vmatrix} (\rho\omega_1)e^{\rho\omega_1 x}[1] & \cdot\ \cdot\ \cdot & (\rho\omega_n)e^{\rho\omega_n(x-1)}[1] & 0 \\ [\alpha_1\omega_1^{k_1}] & \cdot\ \cdot\ \cdot & [\beta_1\omega_n^{k_1}] & D_1 \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ [\alpha_n\omega_1^{k_n}] & \cdot\ \cdot\ \cdot & [\beta_n\omega_n^{k_n}] & D_n \end{vmatrix}^{[5]},$$

$$D_i \equiv \sum_{j=1}^{\mu} e^{\rho\omega_j(1-y)}[\beta_i\omega_j^{k_i+1}] - \sum_{j=\mu+1}^{n} e^{-\rho\omega_j y}[\alpha_i\omega_j^{k_i+1}],$$

and $\bar{\Delta}_1^{(1)}$ is a similar form involving $\bar{\alpha}_j$, $\bar{\beta}_j$, $\bar{k}_j$. On expanding each determinant according to the elements of its first row and collecting terms in (4) we obtain the last line of (3). Hence we may write

$$E_j = \frac{M_j[1]}{[\theta_1]e^{\rho\omega_\mu}+[\theta_0]}, \qquad \bar{E}_j = \frac{\bar{M}_j[1]}{[\bar{\theta}_1]e^{\rho\omega_\mu}+[\bar{\theta}_0]},$$

where M_j, $\bar{M}_j$ are the cofactors of the j-th elements in the first rows of $\Delta_1^{(1)}$, $\bar{\Delta}_1^{(1)}$ respectively. The effect of imposing hypothesis A, under which $\alpha_j = \bar{\alpha}_j$, $\beta_j = \bar{\beta}_j$, $k_j = \bar{k}_j$, is to make $\theta_0 = \bar{\theta}_0$, $\theta_1 = \bar{\theta}_1$, so that E_j and $\bar{E}_j$ may be viewed as two similar forms involving certain bounded exponentials in y, with coefficients in which the leading asymptotic terms are identical.[6] Hence, under our hypothesis, $E_j - \bar{E}_j = [0] = 0(1)/\rho$, and therefore

$$n\rho^{n-1}\left(\frac{\partial G}{\partial x}-\frac{\partial \bar{G}}{\partial x}\right) = 0(1)$$

when ρ is on γ'.

The argument is exactly similar for the case ρ on γ''; the only changes required are in the summations of the last line of (3) where the ranges now must be $(1, \mu - 1)$, (μ, n). Hence for ρ on γ, and therefore also for ρ on Γ, we have

$$n\rho^{n-1}\left(\frac{\partial G}{\partial x}-\frac{\partial \bar{G}}{\partial x}\right) = 0(1),$$

[5] For a more explicit description of $\Delta_1^{(1)}$ see (S), p. 745 and p. 717.

where

[6] See (S), p. 729.

from which we obtain

$$\int_\Gamma n\rho^{n-1}\left(\frac{\partial G}{\partial x}-\frac{\partial \bar{G}}{\partial x}\right)d\rho = 0(N).$$

This result holds uniformly on $0 \leqq x \leqq 1$.

The case when $n = 2\mu$ may be treated in an entirely analogous manner.

With the help of Lemma 2 we can now establish a theorem concerning the behaviour of $Q(a', b')$ and the related function $\bar{Q}(a', b')$, associated with system (2), when $a' \to 0$ or $b' \to 1$.

THEOREM I. *Under hypothesis A, for fixed b' the functions $Q(a', b')$, $\bar{Q}(a', b')$ are either both bounded as $a' \to 0$, or both become infinite in such a way as to have the same asymptotic behaviour: $Q/\bar{Q} \to 1$ as $a' \to 0$. Similar remarks may be made about the behaviour of Q, $\bar{Q}$ as $b' \to 1$.*

From Lemma 2 we have

$$S'_N - \bar{S}'_N = \int_0^1 S_N(y) \int_\Gamma n\rho^{n-1}\left(\frac{\partial G}{\partial x}-\frac{\partial \bar{G}}{\partial x}\right)d\rho dy = 0(N)$$

uniformly on $0 \leqq x \leqq 1$. Hence $S'_N = \bar{S}'_N + 0(N)$, and, the constant Q being the "best" constant, it follows that for some value $x = \xi$ and for some value of N,

$$(Q-1)N \leqq | S'_N(\xi) | = | \bar{S}'_N(\xi) + 0(N) | \leqq \bar{Q}N + | 0(N) |.$$

On adding N to both sides and dividing by N we obtain $Q \leqq \bar{Q} + | 0(1) |$. Likewise we can show that $\bar{Q} \leqq Q + | 0(1) |$. From these results it follows that

$$| Q - \bar{Q} | \leqq K,$$

where K is a constant independent of N and also of a', b'. The conclusions of the theorem are implied in this last result.

Theorem I and Lemma 1 enable us to deduce a useful

COROLLARY. *Under hypothesis A a necessary and sufficient condition that S_N obey Bernstein's theorem on the whole interval $0 \leqq x \leqq 1$ is that $\bar{S}_N$ obey that theorem on the same interval.*

Lemma 1 implies that $S_N - \bar{S}_N = 0(1)$, or, since $| S_N | \leqq 1$, that $| \bar{S}_N | = 0(1)$, uniformly on $0 \leqq x \leqq 1$. On the other hand Theorem I shows that $\bar{Q}$ must be bounded if Q is to be bounded as $a' \to 0$ and $b' \to 1$. But this means that $\bar{S}_N$ obeys Bernstein's theorem on $(0, 1)$.

To illustrate the usefulness of the corollary we shall now give two applications:

I. The Sturm-Liouville case.[7] Let (1) be identified with the Sturm-Liouville system

$$\begin{aligned} u'' \quad &+ (\lambda + l(x))u = 0, \\ u'(0) &- hu(0) = 0, \\ u'(1) &+ Hu(1) = 0. \end{aligned}$$

and (2) with the system

$$\begin{aligned} u'' \quad &+ \lambda u = 0, \\ u'(0) &= u'(1) = 0. \end{aligned}$$

Both systems are normalized and regular and both satisfy the general requirements set forth in the first paragraph. Furthermore, hypothesis A is satisfied, for

$$\begin{aligned} \alpha_1 = \beta_2 = 1, \qquad \beta_1 = \alpha_2 = 0, \qquad k_1 = k_2 = 1, \\ \bar\alpha_1 = \bar\beta_2 = 1, \qquad \bar\beta_1 = \bar\alpha_2 = 0, \qquad \bar k_1 = \bar k_2 = 1. \end{aligned}$$

But system (2) gives rise to sums $\bar S_N$ which are cosine sums on the half period $(0, 1)$, and these in turn, being even trigonometric functions, obey Bernstein's theorem on that half period. Hence, by our corollary, there exists a constant Q such that $| S'_N | \leqq QNL$ uniformly on $0 \leqq x \leqq 1$.

II. A general n-th order case. Let (1) be identified with the normalized and regular system

$$\begin{aligned} L(u) &\equiv u^{(n)} + P_2(x)u^{(n-2)} + \cdots + P_n(x)u + \lambda u = 0, \\ W_j(u) &\equiv u^{(n-j)}(0) - u^{(n-j)}(1) + \sum_{i=0}^{n-j-1} (\alpha_{ij}u^{(i)}(0) + \beta_{ij}u^{(i)}(1)) = 0, \\ &\qquad\qquad (j = 1, 2, \cdots, n), \end{aligned}$$

and (2) with the normalized and regular Fourier system

$$\begin{aligned} u^{(n)} + \lambda u &= 0, \\ u^{(n-j)}(0) - u^{(n-j)}(1) &= 0, \qquad (j = 1, 2, \cdots, n), \end{aligned}$$

both satisfying the further requirement set forth in paragraph 1 concerning the poles of the Green's functions.[8] Hypothesis A is satisfied, for $\alpha_j = \bar\alpha_j = 1$,

[7] The Bernstein extension to this case was proved by Miss E. Carlson, using different methods, in a paper: "Extension of Bernstein's theorem to Sturm-Liouville sums," *Transactions of the American Mathematical Society*, vol. 26 (1924), pp. 230-240.

[8] When n is even the characteristic values of the Fourier system appear as double roots of the characteristic equation, but these give rise to simple poles of the Green's function.

$\beta_j = \bar{\beta}_j = -1$, $k_j = \bar{k}_j = n - j$. But the sums $\bar{S}_N$ are trigonometric polynomials on the period interval $(0, 1)$, and as such obey Bernstein's theorem on that interval. Hence, in this case also, there exists a constant Q such that $| S'_N | \leqq QNL$ uniformly on $0 \leqq x \leqq 1$.

The importance of Theorem I for the study of the behaviour of $Q(a', b')$ as $a' \to 0$ or $b' \to 1$ lies in the fact that it allows us to carry out the study for the function $\bar{Q}(a', b')$ associated with a simpler type of system than we have in (1). Throughout the rest of the paper we shall identify (2) with the system

$$\bar{L}(u) \equiv u^{(n)} + \lambda u = 0,$$
$$\bar{W}_j(u) \equiv \alpha_j u^{(k_j)}(0) + \beta_j u^{(k_j)}(1) = 0, \qquad (j = 1, \cdots, n).$$

Obviously then hypothesis A is satisfied. We now proceed to obtain an explicit formula for the sum $\bar{S}'_N(x)$ when $x = 0$ (and by analogy when $x = 1$ also). It is necessary to consider separately the cases of odd and even order.

Case 1. $n = 2\mu - 1$. The function $S'_N(x)$ is given by

$$\bar{S}'_N(x) = \frac{1}{2\pi i} \int_0^1 S_N(y) \int_\Gamma n\rho^{n-1} \left\{ \frac{\partial \bar{G}}{\partial x} ; \frac{\partial \bar{G}}{\partial x} \right\} d\rho dy, ^{9}$$

where Γ is an arc on the circle $| \rho | = R$, contained on two adjacent sectors which, for definiteness, may be taken to be

$$S_1 : \qquad 0 \leqq \arg \rho \leqq \pi/n,$$
$$S_2 : -\pi/n \leqq \arg \rho \leqq 0.$$

The two arcs associated with S_1, S_2 we shall denote by γ_1, γ_2, and the two halves of each of these by γ'_1, γ''_1, and γ'_2, γ''_2, respectively. In particular, the arcs γ'_1, γ'_2 are those on which the real part of $\rho\omega_\mu \leqq 0$, whereas the arcs γ''_1, γ''_2 are those on which the real part $\geqq 0$.

From (S, p. 745) we have, on putting $x = 0$ and taking $k = 1$,

$$n\rho^{n-1} \left\{ ; \frac{\partial \bar{G}}{\partial x} \right\} = \left\{ ; F^1{}_{11}(0, y, \rho) + \sum_{j=\mu+1}^{n} e^{-\rho\omega_j y} \frac{m_j(0, y, \rho)}{\rho} \right\} + \frac{\Delta_1{}^{(1)}(0, y, \rho)}{[\theta_0] + e^{\rho\omega_\mu}[\theta_1]} .$$

But, from (M, p. 300),

$$F^1{}_{11}(0, y, \rho) \equiv \sum_{j=\mu+1}^{n} \omega_j e^{-\rho\omega_j y}(A_{11}(x) + B_1(y) + \rho\omega_j),$$

whereas, by using an argument similar to that given in (S, p. 732), we obtain for the last term the formula

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[9] The notation $\{A ; B\}$ is used to indicate that A is to be taken when $x \geqq y$, and B when $x \leqq y$.

$$\frac{\Delta_1{}^{(1)}(0, y, \rho)}{[\theta_0] + e^{\rho\omega_\mu}[\theta_1]} = \left(\frac{1}{\theta_0}\delta_1(0, y, \rho) + e^{\rho\omega_\mu}M'(0, y, \rho)\right)_\rho$$

where

$$\delta_1(0, y, \rho) = \begin{vmatrix} \omega_1 \cdots \omega_\mu \omega_{\mu+1}e^{-\rho\omega_{\mu+1}} & \cdots \omega_n e^{-\rho\omega_n} & 0 \\ \alpha_1\omega_1{}^{k_1} \cdots \alpha_1\omega_\mu{}^{k_1}\beta_1\omega_{\mu+1}{}^{k_1} & \cdots \beta_1\omega_n{}^{k_1} & \zeta_1 \\ \vdots \quad \cdots \quad \vdots \quad \vdots & \cdots \quad \vdots & \vdots \\ \alpha_n\omega_1{}^{k_n} \cdots \alpha_n\omega_\mu{}^{k_n}\beta_n\omega_{\mu+1}{}^{k_n} & \cdots \beta_n\omega_n{}^{k_n} & \zeta_n \end{vmatrix}$$

and

$$\zeta_i(y, \rho) = \sum_{j=1}^{\mu} e^{\rho\omega_j(1-y)}\beta_i\omega_j{}^{k_i+1} - \sum_{j=\mu+1}^{n} e^{-\rho\omega_j y}\alpha_i\omega_j{}^{k_i+1}.$$

On substituting these values into the integral

$$I'_1 = \int_0^1 S_N(y) \int_{\gamma'_1} n\rho^{n-1} \left\{ \; ; \; \frac{\partial \bar{G}(0, y, \rho^n)}{\partial x} \right\} d\rho dy,$$

and expanding the result sufficiently, we obtain a large number of separate integrals. These latter, however, are all either $0(1)$ or $0(N)$, except those involving $e^{\rho\omega_\mu(1-y)}$. To verify this statement we note that the integrals in question may be identified with the following typical forms:

$$\text{(i)} \quad \int_0^1 S_N(y) \int_{\gamma'_1} \omega_j{}^2 e^{-\rho\omega_j y}\rho d\rho dy = 0(N), \qquad j = \mu + 1, \cdots, n;$$

$$\text{(ii)} \quad \int_0^1 S_N(y) \int_{\gamma'_1} e^{-\rho\omega_j y}(m_j/\rho) d\rho dy = 0(1), \qquad j = \mu + 1, \cdots, n;$$

$$\text{(iii)} \quad \int_0^1 S_N(y) \int_{\gamma'_1} e^{\rho\omega_\mu}M'\rho d\rho dy = 0(N);$$

$$\text{(iv)} \quad \int_0^1 S_N(y) \int_{\gamma'_1} e^{-\rho\omega_i}m\rho d\rho dy = 0(N), \qquad j = \mu + 1, \cdots, n;$$

$$\text{(v)} \quad \int_0^1 S_N(y) \int_{\gamma'_1} e^{\rho\omega_j(1-y)}\rho d\rho dy = 0(N), \qquad j = 1, \cdots, \mu - 1.$$

(i) may be proved as follows: Let $\rho\omega_j = Re^{i\theta}$; then, when $j = \mu + 1, \cdots, n$, θ will vary over a range (θ_1, θ_2) such that $-\pi/2 < \theta_1, \theta_2 < \pi/2$. Then

$$\left| \int_0^1 S_N(y) \int_{\gamma'_1} \omega_j{}^2 e^{-\rho\omega_j y}\rho d\rho dy \right| \leqq \int_0^1 \int_{\theta_1}^{\theta_2} R^2 e^{-Ry\cos\theta}\, d\theta dy$$
$$\leqq R\int_{\theta_1}^{\theta_2}(1 - e^{-R\cos\theta}) \frac{d\theta}{|\cos\theta|} \leqq 2R \int_{\theta_1}^{\theta_2} \frac{d\theta}{|\cos\theta|} = 0(R) = 0(N).$$

(ii) is implied by (S, p. 714, Lemma III). The left-hand member of (iii) may be written

$$\int_0^1 S_N(y) \int_{\gamma'_1} e^{\rho\omega_\mu}M'\rho d\rho dy = R\int_0^1 S_N(y) \int_{\gamma'_1} e^{\rho\omega_\mu}M'' d\rho dy,$$

where $M'' = M'(\rho/R)$, and hence, by (S, p. 732), it is $= R0(1) = 0(N)$. (iv) may be expressed in terms of (iii), since $e^{-\rho\omega_i} = e^{\rho\omega_\mu}0(1)$, $i = \mu + 1, \cdots, n$, and (v) is similar to (i).

As a consequence of the observations which have just been made we see that

$$I'_1 = \frac{B}{\omega_\mu \theta_0} \int_0^1 S_N(y) \int_{\gamma'_1} \omega_\mu{}^2 e^{\rho\omega_\mu(1-y)} \rho d\rho dy + 0(N),$$

where

$$B = \begin{vmatrix} \omega_1, \cdots \omega_\mu & 0 & \cdots 0 & 0 \\ \alpha_1\omega_1{}^{k_1} \cdots \alpha_1\omega_\mu{}^{k_1} & \beta_1\omega_{\mu+1}{}^{k_1} & \cdots \beta_1\omega_n{}^{k_1} & \beta_1\omega_\mu{}^{k_1} \\ \vdots \quad\quad \vdots & \vdots & \vdots & \vdots \\ \alpha_n\omega_1{}^{k_n} \cdots \alpha_n\omega_\mu{}^{k_n} & \beta_n\omega_{\mu+1}{}^{k_n} & \cdots \beta_n\omega_n{}^{k_n} & \beta_n\omega_\mu{}^{k_n} \end{vmatrix}$$

and θ_0 = the minor of the last element of the first row of B.

The corresponding result for the case when ρ is on γ''_1 may be worked in a similar manner. It is found to be

$$I''_1 = \left(1 - \frac{A}{\omega_\mu \theta_1}\right) \int_0^1 S_N(y) \int_{\gamma''_1} \omega_\mu{}^2 e^{-\rho\omega_\mu y} \rho d\rho dy + 0(N),$$

where

$$A = \begin{vmatrix} \omega_1 \cdots \omega_{\mu-1} & 0 & \cdots 0 & 0 \\ \alpha_1\omega_1{}^{k_1} \cdots \alpha_1\omega_{\mu-1}{}^{k_1} & \beta_1\omega_\mu{}^{k_1} & \cdots \beta_1\omega_n{}^{k_1} & \alpha_1\omega_\mu{}^{k_1} \\ \vdots \quad\quad \vdots & \vdots & \vdots & \vdots \\ \alpha_n\omega_1{}^{k_n} \cdots \alpha_n\omega_{\mu-1}{}^{k_n} & \beta_n\omega_\mu{}^{k_n} & \cdots \beta_n\omega_n{}^{k_n} & \alpha_n\omega_\mu{}^{k_n} \end{vmatrix}$$

and θ_1 = the minor of the last element of the first row of A.

Moreover, to carry over these results from the sector S_1 to the sector S_2 it is only necessary to redistribute the subscripts on the ω's. Using "primes" to indicate the results for the sector S_2, we then obtain formulas I'_2, I''_2 exactly identical with those for I'_1, I''_1 except for the replacement of $\omega_1, \omega_2, \cdots, \omega_n$ by $\omega'_1, \omega'_2, \cdots, \omega'_n$. Furthermore, with our choice of the sectors S_1, S_2 it is easily seen that the sequence $\omega'_1, \omega'_2, \cdots, \omega'_n$ is to be identified with the sequence $\omega_1, \omega_3, \omega_2, \omega_5, \omega_4, \cdots, \omega_n, \omega_{n-1}$.

We now observe

LEMMA 3. (a) $\displaystyle\int_{\gamma'_1} \omega_\mu{}^2 e^{\rho\omega_\mu(1-y)} \rho d\rho + \int_{\gamma'_2} \omega'_\mu{}^2 e^{\rho\omega'_\mu(1-y)} \rho d\rho = 0(N)$;

(b) $\displaystyle\int_{\gamma''_1} \omega_\mu{}^2 e^{-\rho\omega_\mu y} \rho d\rho + \int_{\gamma''_2} \omega'_\mu{}^2 e^{-\rho\omega'_\mu y} \rho d\rho = 0(N)$.

It will be sufficient to prove part (a). Consider the two arcs on the circle $|z| = R$, in the complex z-plane, defined as follows:

$$c_1 : \pi/2 \leqq \arg z \leqq \pi/2 + \phi,$$
$$c_2 : 3\pi/2 - \phi \leqq \arg z \leqq 3\pi/2,$$

where $\phi = \pi/(2n)$. The two integrals in (a) above may now be written

$$\int_{c_1} ze^{z(1-y)}dz + \int_{c_2} ze^{z(1-y)}dz.$$

Let c_3 be the arc of the circle $|z| = R$ defined by $\pi/2 + \phi \leqq \arg z \leqq 3\pi/2 - \phi$, and let c_4 be that diameter of the circle which coincides with imaginary axis. Then, on applying Cauchy's integral theorem, we have

$$\int_{c_1} ze^{z(1-y)}dz + \int_{c_2} ze^{z(1-y)}dz = -\int_{c_3} ze^{z(1-y)}dz - \int_{c_4} ze^{z(1-y)}dz.$$

But

$$\int_{c_3} ze^{z(1-y)}dz = \frac{z_1e^{z_1(1-y)} - z_2e^{z_2(1-y)}}{1-y} - \frac{e^{z_1(1-y)} - e^{z_2(1-y)}}{(1-y)^2}$$
$$= \frac{2iR \cos [2R(1-y)\cos\phi + \phi]e^{-R(1-y)\sin\phi}}{1-y} - \frac{2iR\sin[R(1-y)\cos\phi]e^{-R(1-y)\sin\phi}}{(1-y)^2}$$

where $z_1 = R(-\sin\phi + i\cos\phi)$, $z_2 = R(-\sin\phi - i\cos\phi)$ are the extremities of the arc c_3. When $y \neq 1$ this expression is easily seen to be $0(R) = 0(N)$ as $R \to \infty$, whereas when $y \to 1$ the expression converges to zero for any given value of R. Hence

$$\int_{c_3} ze^{z(1-y)}dz = 0(N).$$

Similarly,

$$\int_{c_4} ze^{z(1-y)}dz = \int_{-R}^{R} (it)e^{it(1-y)}(idt)$$
$$= \frac{-2R\cos R(1-y)}{1-y} + \frac{2\sin R(1-y)}{(1-y)^2} = 0(N),$$

and thus part (a) of the lemma is established.

On adding the integrals I'_1, I''_1, I'_2, I''_2, and eliminating the integrals involving ω'_μ with the help of Lemma 3, we obtain

$$(5) \qquad 2\pi i\bar{S}'_N(0) = \left(\frac{B}{\omega_\mu\theta_0} - \frac{B'}{\omega'_\mu\theta'_0}\right)\int_0^1 S_N(y)\int_{\gamma'_1} \omega_\mu{}^2 e^{\rho\omega_\mu(1-y)}\rho d\rho dy$$
$$-\left(\frac{A}{\omega_\mu\theta_1} - \frac{A'}{\omega'_\mu\theta'_1}\right)\int_0^1 S_N(y)\int_{\gamma''_1} \omega_\mu{}^2 e^{-\rho\omega_\mu y}\rho d\rho dy + 0(N).$$

Case 2. $n = 2\mu$. The treatment in this case is entirely analogous to that of the foregoing case. We shall merely state the results:

$$(6) \qquad 2\pi i\bar{S}'_N(0) = \left(\frac{B}{\omega_\mu\theta_2} - \frac{B'}{\omega'_\mu\theta'_2}\right)\int_0^1 S_N(y)\int_{\gamma_1}\omega_\mu{}^2 e^{\rho\omega_\mu(1-y)}\rho d\rho dy$$
$$-\left(\frac{A}{\omega_{\mu+1}\theta_2} - \frac{A'}{\omega'_{\mu+1}\theta'_2}\right)\int_0^1 S_N(y)\int_{\gamma_2}\omega^2{}_{\mu+1}e^{-\rho\omega_{\mu+1}y}\rho d\rho dy + 0(N),$$

where B is identical with the determinant B of case 1, θ_2 is identical with θ_0 of case 1, A is the same as B except that in the last column the elements are $0, \alpha_1\omega^{k_1}{}_{\mu+1}, \cdots, \alpha_n\omega^{k_n}{}_{\mu+1}$, and B', A' and θ'_2 are the corresponding forms involving $\omega'_1, \omega'_2, \cdots, \omega'_n$.

The integrals remaining in (5) and (6) are definitely $0(N^2)$. Hence, for a fixed value of $b' < 1$, the functions $\bar{Q}(a', b')$, $Q(a', b')$ are bounded as $a' \to 0$ if, and only if, the coefficients of these integrals vanish. Thus we are led to state

THEOREM II. *The necessary and sufficient conditions under which the sums $S_N(x)$ associated with system* (1) *obey Bernstein's theorem on the interval $0 \leqq x \leqq b' < 1$ are as follows:*

(i) *when $n = 2\mu - 1$,*

$$\frac{B}{\omega_\mu\theta_0} - \frac{B'}{\omega'_\mu\theta'_0} = 0, \qquad \frac{A}{\omega_\mu\theta_1} - \frac{A'}{\omega'_\mu\theta'_1} = 0;$$

(ii) *when $n = 2\mu$,*

$$\frac{B}{\omega_\mu\theta_2} - \frac{B'}{\omega'_\mu\theta'_2} = 0, \qquad \frac{A}{\omega_{\mu+1}\theta_2} - \frac{A'}{\omega'_{\mu+1}\theta'_2} = 0.$$

Analogous results may be worked out for the case when $x = 1$.

Theorem II may be applied directly to the Fourier or Sturm-Liouville systems to establish the extension of the Bernstein theorem to the point $x = 0$. On the other hand, in the case of the system $u''' + \lambda u = 0$, $u''(0) = 0$, $u''(1) = 0$, $u'(0) + u'(1) = 0$, for example, the theorem shows that the extension to the point $x = 0$ is not possible.

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ON THE LINEARITY OF PENCILS OF CURVES ON ALGEBRAIC SURFACES.*

By O. F. G. Schilling [1] and O. Zariski.

The object of this note is to give an arithmetical proof of the following often used theorem: "*If a pencil of curves on an algebraic surface has a base point at a simple point of the surface then the pencil is either a linear system or its curves are cut out by hypersurfaces* $(\phi + \lambda\psi)^{\rho}$." The essential feature of our approach consists in eliminating the difficulties which arise from the possible singularities of the curves at the simple base point. The interpretation of the pencil of curves as a rational transform of the given surface allows us to apply a theorem proved by one of us.[2]

Let K be a field of algebraic functions of two variables over an algebraically closed field k. An algebraic surface f in the affine n-dimensional space S_n over k is said to be a model of K in S_n if the quotient field of $k[x_1, \cdots, x_n]/\mathfrak{p}(f)$ which is determined by the prime ideal $\mathfrak{p}(f)$ defining the surface f, is isomorphic with K. Thus f is described by the order $k[\xi_1, \cdots, \xi_n] = \mathfrak{O}$ in K where $\xi_i = x_i \bmod \mathfrak{p}(f)$. The 0-dimensional prime ideals $\mathfrak{p}$ of $k[x_1, \cdots, x_n]$ which divide $\mathfrak{p}(f)$ correspond to the points P of the surface f. A point P with the coördinates $\{a_1, \cdots, a_n\}$ is called a simple point of f if

(i) the ideal $(\xi_1 - a_1, \cdots, \xi_n - a_n)$ is a 0-dimensional prime ideal $\mathfrak{p}$ in the integral closure of $\mathfrak{O}$ and

(ii) it is possible to choose two algebraically independent elements, say ξ_1, ξ_2 among $\xi_1, \xi_2, \cdots, \xi_n$ such that the ideal $(\xi_1 - a_1, \xi_2 - a_2)$ is divisible by $\mathfrak{p}$ but not divisible by any primary ideal belonging to $\mathfrak{p}$.

It can be shown that all elements of $\mathfrak{O}$ can be expanded in formal power series of $u = \xi_1 - a_1$, $v = \xi_2 - a_2$ with coefficients in k.[3] Hence the elements of $\mathfrak{O}$ are contained in the ring of holomorphic functions $\{\sum_{i,j \geqq 0} a_{ij} u^i v^j\}$ which itself is contained in the field of all formal meromorphic functions of u, v:

* Received February 7, 1938.

[1] Johnston Scholar of the Johns Hopkins University for 1937-1938.

[2] O. Zariski, "Polynomial ideals defined by infinitely near base points," § 13, *American Journal of Mathematics*, vol. 60 (1938).

[3] O. Zariski, "Some results in the arithmetic theory of algebraic functions of several variables," *Proceedings of the National Academy of Sciences*, vol. 23 (1937).

$$k\{u, v\} = \{(\Sigma a_{ij}u^iv^j)(\Sigma b_{ij}u^iv^j)^{-1}\}.$$

An irreducible algebraic system Σ_r of curves on the surface f is given by an irreducible algebraic correspondence between f and a r-dimensional algebraic variety V_r such that to a generic point on V_r there corresponds a curve $C \subset \Sigma_r$ on f. A pencil Σ_1 of curves on f is an irreducible algebraic system Σ_1 such that there passes through a generic point P of f exactly one curve C in Σ_1.[4] This definition of a pencil Σ_1 is equivalent to the following: the function field K_1 belonging to the variety V_1 defining Σ_1 is a rational transform of the surface f, i. e. K_1 is isomorphic with a 1-dimensional subfield $\bar{K}_1$ of K.

If V_r is a linear r-dimensional space and if the curves of Σ_r are cut out by hypersurfaces $\phi = \sum_{i=0}^{r} \lambda_i\phi_i = 0$, ϕ_i being forms in the imbedding space of f, then Σ_r is called a linear system. In a linear system one usually omits the fixed curves which are cut out by all hypersurfaces ϕ.

After these preliminary remarks we proceed to the proof of the

THEOREM. *If a pencil of curves Σ_1 on an algebraic surface f has a base point at a simple point of f then Σ_1 is either a linear system or its curves are cut out by hypersurfaces $(\phi + \lambda\psi)^p = 0$.*

Proof. Let P be the simple base point of the pencil Σ_1. Since P is assumed to be a simple point of the surface f, there exist functions u, v in $\mathfrak{O}$ defining a field of meromorphic functions $k\{u, v\}$ which contains a subfield $K^* \cong K$. Moreover, $u = v = 0$ at P. Consequently, the field $\bar{K}_1$ which belongs to the pencil Σ_1 has also an isomorphic map K^*_1 in $k\{u, v\}$. Thus each element $a^* \varepsilon K^*$ is represented by a ratio $\dfrac{\alpha(u, v)}{\beta(u, v)}$ of holomorphic functions $\alpha(u, v)$, $\beta(u, v)$. Moreover, there exists a function $\dfrac{\alpha(u, v)}{\beta(u, v)} = A^* \varepsilon K^*_1$ such that

$$\alpha(0, 0) = \beta(0, 0) = 0$$

when u and v assume the constant values and 0, 0, respectively, at the given point P. The existence of such a function A^* is a consequence of the assumption that P is a simple base point of Σ_1, i. e. that there corresponds to P the whole curve V_1 under the correspondence between f and V_1. In fact, let us assume that the surface f is given in a 3-dimensional affine space $k[x_1, x_2, x_3]$ and that V_1 is given in an n-dimensional projective space $k[y_0, y_1, \cdot\cdot\cdot, y_n]$. Since K_1 is a rational transform of K we have relations of the following type

[4] For these definitions see for example O. Zariski, "Algebraic surfaces," Chapters II, V, *Ergebnisse der Mathematik und ihrer Grenzgebiete* (Berlin, 1935).

$$P(x_1, x_2, x_3)y_i - Q_i(x_1, x_2, x_3)y_0 = 0 \qquad (i = 1, 2, \cdots, n)$$

where $P(x_1, x_2, x_3)$ and $Q_i(x_1, x_2, x_3)$ are polynomials in x_1, x_2, x_3. Using the imbedding of K as K^* in $k\{u, v\}$ we obtain

$$P_i(u, v)y_i - Q_i(u, v)y_0 = 0 \qquad (i = 1, 2, \cdots, n)$$

where $P_i(u, v)$ and $Q_i(u, v)$ are relatively prime holomorphic functions in u, v. These equations can be considered as relations which are contained in the ideal $\mathfrak{C}$ of relations defining the correspondence. The assumption that P be a base point of Σ_1 implies then that

$$P_i(0, 0)y_i - Q_i(0, 0)y_0 \equiv 0 \text{ in } K^*_1.$$

We may suppose that y_0 is different from 0, then

$$P_i(0, 0)\frac{y_i}{y_0} - Q_i(0, 0) \equiv 0.$$

Consequently, since at least one function y_i/y_0 of the field $K^*_1 \cong K_1$ does not lie in k,

$$P_i(0, 0) = Q_i(0, 0) = 0,$$

or y_i/y_0 is a function having the desired properties. Now we are in a position to apply a result of the general theory of valuation ideals stating that for each function $A^* = \dfrac{\alpha(u, v)}{\beta(u, v)}$, where $\alpha(0, 0) = \beta(0, 0) = 0$, and α, β are relatively prime, there exists a prime divisor $\mathfrak{P}$ of $k\{u, v\}$ which maps $k\{u, v\}$ upon a purely transcendental field $k(t)$ in which the map $A^* * \mathfrak{P}$ of A^* is a transcendental quantity with respect to k.[5] Consequently, the field $K^*_1 \cong K_1$ is mapped upon a transcendental subfield of $k(t)$. Hence K^*_1 is itself a purely transcendental subfield, for the divisor $\mathfrak{P}$ acts as an isomorphic mapping on K^*_1, since K^*_1 and its map have the same degree of transcendentality. We have

$$K^*_1 \cong \bar{K}_1 = k(\lambda) \subset K$$

where λ is the ratio $\dfrac{p(x_1, x_2, x_3)}{q(x_1, x_2, x_3)}$ of relatively prime polynomials $p(x_1, x_2, x_3)$ and $q(x_1, x_2, x_3)$ in $k[x_1, x_2, x_3]$.

We observe that we do not change the nature of the algebraic pencil Σ_1 if we use instead of the original variety V_1 the birationally equivalent curve $k[\lambda]$ for the definition of Σ_1.

Now it remains to be shown that Σ_1 is a linear system cut out by surfaces

[5] See note 2, *loc. cit.*, p. 203

$\lambda q(x_1, x_2, x_3) - p(x_1, x_2, x_3) = 0$ or a system of curves cut out by the surfaces $(\lambda q(x_1, x_2, x_3) - p(x_1, x_2, x_3))^\rho = 0$, $\rho > 1$. Consider for this purpose the ideal

$$\mathfrak{A} = (\lambda(x_1, x_2, x_3) - p(x_1, x_2, x_3), f) = (\lambda p - q, f)$$

in the ring $k[x_1, x_2, x_3, \lambda]$ where $(f) = \mathfrak{p}(f)$ denotes the prime ideal defining the surface f. According to a well-known theorem of Macauly the ideal $\mathfrak{A}$ is unmixed of dimension 2, thus

$$\mathfrak{A} = [\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_s]$$

where the ideals $\mathfrak{q}_i$ are 2-dimensional primary ideals with the associated prime ideals $\mathfrak{P}_i$. The contracted ideal $\bar{\mathfrak{A}} = \mathfrak{A} \cap k[x_1, x_2, x_3]$ of $\mathfrak{A}$ is equal to (f). Hence

$$\bar{\mathfrak{A}} = (f) = [\bar{\mathfrak{q}}_1, \bar{\mathfrak{q}}_2, \cdots, \bar{\mathfrak{q}}_s]$$

where $\bar{\mathfrak{q}}_i = \mathfrak{q}_i \cap k[x_1, x_2, x_3]$. This representation implies that one component $\bar{\mathfrak{q}}_i$ must be equal to (f); let $\bar{\mathfrak{q}}_1$ be such a component.

We consider next an arbitrary element $F(x_1, x_2, x_3, \lambda)$ lying in $\mathfrak{q}_1$, then

$$q^\sigma F(x_1, x_2, x_3, \lambda) = A(x_1, x_2, x_3, \lambda)(\lambda q - p) + B(x_1, x_2, x_3).$$

Since $F(x_1, x_2, x_3, \lambda)$ and $\lambda p - q$ both lie in $\mathfrak{q}_1$ we get

$$B(x_1, x_2, x_3) \subset \mathfrak{q}_1,$$

consequently

$$\begin{aligned} B(x_1, x_2, x_3) &\equiv 0 \pmod{\bar{\mathfrak{q}}_1} \\ &\equiv 0 \pmod{f}. \end{aligned}$$

There therefore exists a common exponent $\tau > 0$ such that

$$q^\tau F(x_1, x_2, x_3, \lambda) \equiv 0 \pmod{\mathfrak{A}}$$

for any element $F \subset \mathfrak{q}_1$, because $\mathfrak{q}_1$ has a finite base. Hence

$$q^\tau \mathfrak{q}_1 \equiv 0 \pmod{\mathfrak{q}_2},$$

consequently

$$q^\tau \equiv 0 \pmod{\mathfrak{q}_2},$$

for $\mathfrak{A}$ is an unmixed ideal and consequently all components $\mathfrak{q}_i$ have the same dimension. Therefore

$$q \equiv 0 \pmod{\mathfrak{P}_2}$$

and also

$$q \equiv 0 \pmod{\bar{\mathfrak{P}}_2}.$$

Since $\lambda = p/q$ we have $q \not\equiv 0(f)$, thus

$$\bar{\mathfrak{q}}_2 \not\equiv 0(f) \quad \text{and} \quad \bar{\mathfrak{P}}_2 \not\equiv 0(f).$$

Therefore $\bar{\mathfrak{q}}_2 \supset (f)$, consequently the ideals $\bar{\mathfrak{q}}_2, \cdots, \bar{\mathfrak{q}}_3$ are 0- or 1-dimensional ideals. They must be 1-dimensional, for $\bar{\mathfrak{q}}_2 k[x_1, x_2, x_3, \lambda] \subset \mathfrak{q}_2$ and hence

$$\begin{aligned} \dim \mathfrak{q}_2 = 2 &\leq \dim \bar{\mathfrak{q}}_2 k[x_1, x_2, x_3, \lambda] \\ &= \dim \bar{\mathfrak{q}}_2 + 1, \end{aligned}$$

or

$$\dim \bar{\mathfrak{q}}_2 = 1.$$

This relation between the dimensions of $\mathfrak{q}_2$ and $\bar{\mathfrak{q}}_2$ shows that the components $\bar{\mathfrak{q}}_2, \cdots, \bar{\mathfrak{q}}_s$ do not depend on λ, i. e. they are extended ideals of $\bar{\mathfrak{q}}_2, \cdots, \bar{\mathfrak{q}}_s$. In geometric terms, the curves $\mathfrak{q}_2, \cdots, \mathfrak{q}_s$ correspond to the entire line $k[\lambda]$ under the algebraic correspondence. Such fixed components shall be left out in the definition of a linear system, and consequently $\mathfrak{q}_1 = \mathfrak{q}$ is the ideal defining Σ_1. According to the properties of primary ideals we have

$$(\lambda q - p)^\rho \subset \mathfrak{q},$$

i. e. Σ_1 is cut out by the hypersurfaces $(\lambda q - p)^\rho = 0$ where fixed components are omitted.

We remark that if q lies in one of the components $\mathfrak{q}_i \neq \mathfrak{q}$ then also $p \subset \mathfrak{q}_i$ for $\lambda q - p \subset \mathfrak{A}$; hence $p = q = 0$ occur among the equations defining the correspondence $\mathfrak{q}$.

THE JOHNS HOPKINS UNIVERSITY.

SOME SINGULAR PROPERTIES OF CONFORMAL TRANSFORMATIONS BETWEEN RIEMANNIAN SPACES.*

By Virginia Modesitt.

1. Introduction. Two Riemannian spaces V_n and V'_n are in conformal correspondence if their fundamental tensors are related by [1]

$$g'_{ij} = e^{2\sigma} g_{ij} \qquad (i, j = 1, \cdots n), \tag{1.1}$$

where σ is any function of the x's.[2] The purpose of this paper is to investigate some geometric properties of corresponding curves and subspaces in V_n and V'_n.

From (1.1), it follows that

$$g'^{ij} = e^{-2\sigma} g^{ij}; \tag{1.2}$$

that contravariant and covariant components of corresponding unit vectors are related by

$$\lambda'^i = e^{-\sigma}\lambda^i, \qquad \lambda'_i = e^{\sigma}\lambda_i; \tag{1.3}$$

and that Christoffel symbols for the two spaces are given by

$$\begin{aligned} [ij, k]' &= e^{2\sigma}([ij, k] + g_{ik}\bar{\sigma}_j + g_{jk}\bar{\sigma}_i - g_{ij}\bar{\sigma}_k), \\ \left\{ {i \atop jk} \right\}' &= \left\{ {i \atop jk} \right\} + \delta_j{}^i\bar{\sigma}_k + \delta_k{}^i\bar{\sigma}_j - g_{jk}\bar{\sigma}^i, \end{aligned} \tag{1.4}$$

where $\bar{\sigma}_i = \partial\sigma/\partial x^i$, $\bar{\sigma}^i = g^{im}\bar{\sigma}_m$. The bar is used throughout to indicate that the components of a vector are not necessarily unit. It is to be noted that σ^i are the components of the congruence of curves normal to the family of hypersurfaces, $\sigma = \text{constant}$. We shall call this congruence the congruence of σ-curves and the family of hypersurfaces H_σ.

It will also be useful to have the relation between the covariant derivatives of unit contravariant components of corresponding directions. From (1.3) and (1.4), we obtain

$$\lambda'^i{}_{,j} = e^{-\sigma}(\lambda^i{}_{,j} + \delta_j{}^i\bar{\sigma}_k\lambda^k - g_{jk}\lambda^k\bar{\sigma}^i), \tag{1.5}$$

* Received July 17, 1937.

[1] L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1926, p. 89. We shall refer to this book as R. G.

[2] We shall assume in what follows that the fundamental forms of V_n and V'_n are positive definite and that the function σ is not identically a constant.

where the dot is used to denote covariant differentiation with respect to the g''s and the comma to denote covariant differentiation with respect to the g's.

2. Curves with corresponding principal normals. The p-th normals of a curve C are given by the Frenet formulas [3]

$$(2.1)\qquad {}_{p+1}\lambda^i = \rho_p({}_1\lambda^j\, {}_p\lambda^i{}_{,j} + (1/\rho_{p-1})\, {}_{p-1}\lambda^i) \qquad \begin{array}{c}(p = 1,\cdots, n-1)\\ (1/\rho_0 = 1/\rho_n = 0),\end{array}$$

where the p-th curvatures of C satisfy the conditions

$$(2.2)\qquad 1/\rho_p = {}^{p+1}\lambda_i\, {}_1\lambda^j\, {}_p\lambda^i{}_{,j}. \qquad (p = 1,\cdots, n-1).$$

Similar relations in the primed quantities may be written for the normals and curvatures of the corresponding curve C'.

In particular, the principal normals of C' are given by

$$(2.3)\qquad {}_2\lambda'^i = \rho'_1\, {}_1\lambda'^j\, {}_1\lambda'^i{}_{.j}.$$

Since the tangent vectors of C and C' are in corresponding directions, their unit components are related by ${}_1\lambda'^i = e^{-\sigma}\, {}_1\lambda^i$. By means of (1.5), it follows from (2.3) that principal normals of C and C' are in the relation

$$(2.4)\qquad {}_2\lambda'^i/\rho'_1 = e^{-2\sigma}({}_2\lambda^i/\rho_1 + \bar\sigma_k\, {}_1\lambda^k\, {}_1\lambda^i - \bar\sigma^i).$$

Hence, in general, the principal normals of C and C' are not in corresponding directions. If their directions do correspond, the vector, $\bar\sigma_k\, {}_1\lambda^k\, {}_1\lambda^i - \bar\sigma^i$, must either have zero components or else must be in the direction ${}_2\lambda^i$. In the first case, C is a $\bar\sigma$-curve. We shall discuss such curves more particularly in the next section and confine ourselves here to the second case,—that in which we can write

$$(2.5)\qquad {}_2\lambda^i = a\, {}_1\lambda^i + b\bar\sigma^i.$$

Let us consider the surface generated by the σ-curves at points of a curve and call the V_2 so formed the σ-surface of the curve. From (2.5), it is seen that for a curve C of the type we are considering, the principal normals are directions in the σ-surface of C, and conversely, if ${}_2\lambda^i$ is a direction in the σ-surface of C, then C and C' are curves with corresponding principal normals.

The principal normals of a curve (not geodesic in V_n) which lies in a subspace V_m of V_n, $(m < n)$, may be written [4]

$$(2.6)\qquad {}_2\lambda^i/\rho_1 = \nu^i/\rho_g + \zeta^i/R$$

[3] R. G., p. 106.
[4] R. G., p. 165.

where ν^i are the unit components of the principal normal of the curve in V_m, and ζ^i is the normal curvature vector to V_m along C. $1/\rho_g$ is the first curvature of C in V_m, and $1/R$ is the normal curvature of V_m for the given curve.

Since, for a curve with corresponding principal normals, the directions of these principal normals coincide with the directions of the normals to C in the σ-surface of C, it follows that $1/R$ must be zero, i. e., that such a curve be asymptotic on its σ-surface. Conversely, if V_2 be any surface generated by a one-parameter family of σ-curves, and if C be asymptotic on this surface, then from (2.6) it is seen that ${}_2\lambda^i$ is a direction in the V_2 and hence that C and C' have corresponding principal normals.

If C is geodesic in V_n, it is both geodesic and asymptotic on its σ-surface, V_2. It is found from (2.4) that the principal normals of C' are directions in V'_2 and hence that C' is also asymptotic on its σ-surface. In this case, however, the principal normals of C are indeterminate.

We may now state the theorem: *If the principal normals of C and C' are in corresponding directions, (C is not a σ-curve and is not geodesic in V_n), then C is asymptotic on its σ-surface. Conversely, if C, (not a σ-curve and not geodesic in V_n), is asymptotic on its σ-surface, then the principal normals of C and C' are in corresponding directions.*

Let us assume that any two consecutive normals of two corresponding curves C and C', say the $(p-2)$-nd and $(p-1)$-st, $(p>2)$, are in corresponding directions:

$$
{}_{p-1}\lambda'^i = e^{-\sigma}\,{}_{p-1}\lambda^i, \qquad {}_p\lambda'^i = e^{-\sigma}\,{}_p\lambda^i \qquad (p>2). \tag{2.7}
$$

If we write the formula for the $(p-1)$-st curvature of C' from (2.2) and replace the primed quantities in terms of the unprimed on the right hand side, we will obtain

$$
1/\rho'_{p-1} = e^{-\sigma}(1/\rho_{p-1}). \tag{2.8}
$$

From (2.7), (2.8), (1.5), and the Frenet formula for the p-th normal of C', we obtain

$$
{}_{p+1}\lambda'^i = e^{-2\sigma}\rho'_p((1/\rho_p)\,{}_{p+1}\lambda^i + \bar{\sigma}_k\,{}_p\lambda^k\,{}_1\lambda^i).
$$

Multiplying both sides by ${}^1\lambda_i$, summing on i, gives

$$
\bar{\sigma}_k\,{}_p\lambda^k = 0 \qquad (p>2), \tag{2.9}
$$

whence it follows that

$$
{}_{p+1}\lambda'^i = e^{-\sigma}\,{}_{p+1}\lambda^i, \qquad \rho'_p/\rho_p = e^{\sigma}. \tag{2.10}
$$

If, then, any two consecutive normals, ${}_{p-1}\lambda^i$ and ${}_p\lambda^i$, $(p>2)$, of C and C'

are in corresponding directions, all succeeding normals will also be in corresponding directions and the ratios, ρ'_m/ρ_m, of the m-th radii of curvature of the two curves, $(m = p - 1, \cdots, n)$, will be equal to the coefficient of magnification, e^{σ}.

Similarly, it can be shown that if the p-th and $(p - 1)$-st normals of C and C' are in corresponding directions, then all normals before the $(p - 1)$-st are also in corresponding directions and the radii of curvature of the two curves before the p-th will be in the ratio e^{σ}, with the exception of the radii of first curvature. Since, in particular, we will have ${}_2\lambda'^i = e^{-\sigma}\, {}_2\lambda^i$, C and C' are curves with corresponding principal normals. It follows from (2. 2) that, for such curves, first curvatures are in the relation

$$1/\rho'_1 = e^{-\sigma}(1/\rho_1 - \bar{\sigma}_k\, {}_2\lambda^k). \tag{2.11}$$

If the normal ennuple of C, ${}_1\lambda^i, {}_2\lambda^i, \cdots, {}_n\lambda^i$, be considered as a cyclical set of directions, and hence ${}_n\lambda^i$ and ${}_1\lambda^i$, ${}_1\lambda^i$ and ${}_2\lambda^i$, as pairs of consecutive directions, the above results still hold. Consequently: *If any two consecutive directions of the normal ennuples of C and C' correspond, where the directions, ${}_1\lambda^i, \cdots, {}_n\lambda^i$, are considered as a cyclical set, then the remaining directions for the two curves also correspond, and the ratios, ρ'_p/ρ_p, $(p = 2, \cdots, n - 1)$, of the p-th radii of curvature are equal to the coefficient of magnification, e^{σ}.*

3. Properties of σ-curves. We have defined σ-curves as the congruence of curves orthogonal to the hypersurfaces H_σ. If C is a σ-curve,

$$\bar{\sigma}^i = \tau\, {}_1\lambda^i, \qquad \bar{\sigma}_i = \tau\, {}^1\lambda_i. \tag{3.1}$$

Hence, it follows from (2. 4) that the σ-curves are curves with corresponding principal normals. Equation (2. 11) shows that they differ from the curves with corresponding principal normals already considered in that first curvatures are related by

$$1/\rho'_1 = e^{-\sigma}(1/\rho_1). \tag{3.2}$$

Since the σ-curves fulfill the conditions of the last theorem of section two, the ratios of the p-th radii of curvature of C and C' are also equal to e^{σ}, $(p = 2, \cdots, n - 1)$, and the p-th normals, $(p = 2, \cdots, n)$ are in corresponding directions.

Conversely, if C and C' have corresponding principal normals, we may write, $\bar{\sigma}^i = \alpha\, {}_1\lambda^i + \beta\, {}_2\lambda^i$, and if $1/\rho'_1 = e^{-\sigma}(1/\rho_1)$, it follows that $\beta = 0$ and C is a σ-curve. Hence: *Necessary and sufficient conditions that C be a σ-curve are* 1) *that C and C' have corresponding principal normals, and* 2) *that the ratio of the radii of first curvature of C and C' be equal to the coefficient of magnification, e^{σ}.*

If directions μ^i are parallel along a curve C, we have

$$\mu^i{}_{,j}\, {}_1\lambda^j = 0. \tag{3.3}$$

Writing the same condition for corresponding directions in V'_n, we find by means of (1.5) that

$$\mu'^i{}_{.j}\, {}_1\lambda'^j = e^{-2\sigma}(\mu^i{}_{,j}\, {}_1\lambda^j + \bar{\sigma}_k \mu^k\, {}_1\lambda^i - g_{jk}\, {}_1\lambda^j \mu^k \bar{\sigma}^i). \tag{3.4}$$

It follows from (3.1) and (3.3) that *if directions μ^i are parallel along a σ-curve C, the corresponding directions are parallel along C'.* In particular, if C is geodesic in V_n, from (3.4) for $\mu^i = {}_1\lambda^i$ we may state: *A necessary and sufficient condition that a curve C, geodesic in V_n, correspond to a geodesic in V'_n is that C be a σ-curve.*

4. Osculating spaces of a curve. By the osculating space, O_p, of a curve is meant the V_p determined at a point by the first p directions of the normal ennuple of C. As a special case, it appears, from (2.4), that O'_2 will correspond to O_2 if and only if C is a curve with corresponding principal normals.

It can be shown by induction that ${}_p\lambda'$ can be written as a linear combination of the directions, ${}_1\lambda^i, \cdots, {}_p\lambda^i$, σ^i, ${}_1\eta^i, \cdots, {}_{p-2}\eta^i$, where ${}_k\eta^i = {}_{k-1}\eta^i{}_{,j}\, {}_1\lambda^j$, $(k = 2, \cdots, p-2)$, is the k-th associate direction of σ^i along C. If σ^i, ${}_1\eta^i, \cdots, {}_{p-2}\eta^i$ are directions in the O_p of C, then O_p and O'_p correspond. The converse is also true. We may write the equations

$$\begin{aligned}
{}_2\lambda'^i &= {}_2a\, {}_2\lambda^i + {}_2b\, {}_1\lambda^i + {}_2c\sigma^i, \\
{}_3\lambda'^i &= {}_3a\, {}_3\lambda^i + {}_3b\, {}_2\lambda^i + {}_3c\, {}_1\lambda^i + {}_3d\sigma^i + {}_3e\, {}_1\eta^i, \\
&\cdots\cdots\cdots\cdots\cdots \\
{}_p\lambda'^i &= {}_pa\, {}_p\lambda^i + \cdots + {}_pb\, {}_1\lambda^i + {}_pc\sigma^i + {}_pd\, {}_1\eta^i + \cdots + {}_pe\, {}_{p-2}\eta^i.
\end{aligned} \tag{4.1}$$

If any ${}_k\eta^i$ is a zero vector, then ${}_r\eta^i$, $(r > k)$, are also zero vectors. For $r < k$, the e's are not zero and under the hypothesis that O_p and O'_p correspond, equations (4.1) can be solved for σ^i, ${}_1\eta^i, \cdots, {}_{k-1}\eta^i$ as linear functions of ${}_1\lambda^i, \cdots, {}_p\lambda^i$. Hence: *If σ^i and its $(p-2)$ associate directions (which may or may not be zero vectors) lie in O_p, the osculating spaces of C and C' correspond and conversely.*

5. Properties of curves in corresponding subspaces. Let C and C' be curves in subspaces V_m and V'_m, $(m < n)$, immersed in V_n and V'_n.[5] We shall denote by ${}_\tau\xi^i$, $(\tau = 1, \cdots, n-m)$, a set of $(n-m)$ mutually

[5] R. G., p. 143 ff., gives a detailed account of the geometry of subspaces.

orthogonal unit directions in V_n normal to V_m. Corresponding to each of these normals a tensor ${}^{\tau}\Omega_{\alpha\beta}$ is defined by [6]

$$(5.1) \qquad {}^{\tau}\Omega_{\alpha\beta} = g_{ij}x^i{}_{,\alpha\beta}\,{}_{\tau}\xi^j + [kl, j]_g\, x^k{}_{,\alpha}x^l{}_{,\beta}\,{}_{\tau}\xi^j \qquad (\alpha, \beta = 1 \cdots m)$$

and is related to the corresponding tensor ${}^{\tau}\Omega'_{\alpha\beta}$ formed for the direction ${}_{\tau}\xi'^i$ by

$$(5.2) \qquad {}^{\tau}\Omega'_{\alpha\beta} = e^{\sigma}\,({}^{\tau}\Omega_{\alpha\beta} - a_{\alpha\beta}\bar{\sigma}_k\,{}_{\tau}\xi^k),$$

where $a_{\alpha\beta}$ are the coefficients of the first fundamental form of V_m.

The normal curvature vector of V_m in a given direction is defined by the relation

$$(5.3) \qquad \bar{\zeta}^i = {}^{\tau}\Omega_{\alpha\beta}\,{}_1\lambda^{\alpha}\,{}_1\lambda^{\beta}\,{}_{\tau}\xi^i$$

and is related to the normal curvature vector of V'_m by

$$(5.4) \qquad \zeta'^i/R' = e^{-2\sigma}(\zeta^i/R - \sum_{\tau=1}^{n-m}\bar{\sigma}_k\,{}_{\tau}\xi^k\,{}_{\tau}\xi^i),$$

where $1/R$ is the normal curvature of the V_m for the given direction. Since $\sum_{\tau=1}^{n}\bar{\sigma}_k\,{}_{\tau}\xi^k\,{}_{\tau}\xi^i = \bar{\sigma}_k g^{ki} = \bar{\sigma}^i$, we may also write (5.4) in the form

$$(5.5) \qquad \zeta'^i/R' = e^{-2\sigma}(\zeta^i/R - \bar{\sigma}^i + \bar{\sigma}_k\,{}_1\lambda^k\,{}_1\lambda^i + \bar{\sigma}_k\nu_1{}^k\nu_1{}^i + \cdots + \bar{\sigma}_k\nu^k{}_{m-1}\nu^i{}_{m-1}),$$

where $\nu_1{}^i, \cdots, \nu^i{}_{m-1}$ are the $(m-1)$ normals to C in V_m.

If V_m contain a congruence of σ-curves, from (5.4) it is seen that $\zeta'^i = e^{-\sigma}\zeta^i$. Conversely, if the normal curvature vectors at points of corresponding curves C and C' are in corresponding directions, from the relation (2.6) written for C', we find after replacing primed quantities by means of unprimed that

$$(5.6) \qquad 1/R' = e^{-\sigma}(1/R - \bar{\sigma}_k\zeta^k).$$

Since V_m contains σ-curves, normal curvatures at points of C and C' are related by

$$1/R' = e^{-\sigma}(1/R)$$

and hence asymptotic lines in V_m and V'_m correspond. Conversely, if asymptotic lines C and C' correspond, it follows from (5.4) that the V_m contains a congruence of σ-curves. Accordingly: *A necessary and sufficient condition that asymptotic curves in V_m and V'_m correspond is that V_m contain a congruence of σ-curves.*

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[6] R. G., p. 160.

If a curve C be a line of curvature for the normal ${}_{\tau}\xi^i$ it will satisfy the condition[7]

$$(5.7)\qquad {}_{\tau}\xi^i{}_{,j}\,{}_1\lambda^j = \rho\,{}_1\lambda^i.$$

From (1.5) and (5.7), it follows that the same condition holds for C' and the normal ${}_{\tau}\xi'^i$, i. e., *lines of curvature in V_m for a given normal correspond to lines of curvature in V'_m for the corresponding normal.*

If the normal curvature vector to a V_m (not containing σ-curves) at a point and in a given direction corresponds to the normal curvature vector at the corresponding point and in the corresponding direction, then, from (5.4), by means of (5.6), we have

$$\bar{\sigma}_k\zeta^k\zeta^i = \sum_{\tau=1}^{n-m}\bar{\sigma}_k\,{}_{\tau}\xi^k\,{}_{\tau}\xi^i.$$

Multiplying both sides by any one of the ${}^{\tau}\xi_i$ and summing on i, gives

$$(\bar{\sigma}_k\zeta^k\zeta^i - \bar{\sigma}^i)\,{}^{\tau}\xi_i = 0 \qquad (\tau = 1,\cdots, n-m).$$

This equation will be satisfied if the vector, $\bar{\eta}^i = \bar{\sigma}_k\zeta^k\zeta^i - \bar{\sigma}^i$, lies in V_m or if the direction σ^i coincide with the direction ζ^i. Conversely, if, for a given direction at a point, $\sigma^i = \zeta^i$ or if $\bar{\eta}^i$ lies in V_m, it follows from (5.5) that ζ^i for that direction will correspond to ζ'^i for the corresponding direction in V'_m. *The normal curvature vector to a V_m (not containing σ-curves) at a point and in a given direction will correspond to the normal curvature vector to V'_m at the corresponding point and in the corresponding direction, if and only if the vector $\bar{\eta}^i$ lie in V_m or the vector σ^i coincide with the vector ζ^i for the given direction.*

If normal curvature vectors correspond, it follows from (5.4) and (5.3) that

$${}^{\tau}\Omega_{\alpha\beta}\,{}_1\lambda^{\alpha}\,{}_1\lambda^{\beta}\,{}_{\tau}\xi^i = \rho\sum_{\tau=1}^{n-m}\bar{\sigma}_k\,{}_{\tau}\xi^k\,{}_{\tau}\xi^i$$

or, since the ${}_{\tau}\xi^i$ are linearly independent, that

$$({}^{\tau}\Omega_{\alpha\beta} - \rho\bar{\sigma}_k\,{}_{\tau}\xi^k a_{\alpha\beta})\,{}_1\lambda^{\alpha}\,{}_1\lambda^{\beta} = 0 \qquad (\tau = 1\cdots n-m).$$

This equation says that *if the normal curvature vectors to V_m along C correspond to the normal curvature vectors along C'* FOR EVERY *C then V_m has completely indeterminate lines of curvature for all normals, ${}_{\tau}\xi^i$, and the normal curvatures in these $(n-m)$ directions, defined by*

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[7] R. G., p. 168.

$$\frac{1}{R_\tau} = \frac{{}^\tau\Omega_{\alpha\beta}\,{}_1\lambda^\alpha\,{}_1\lambda^\beta}{a_{\alpha\beta}\,{}_1\lambda^\alpha\,{}_1\lambda_\beta},$$

are proportional to the cosines of the angles which the normals make with the σ-curves, or else V_m contains a congruence of σ-curves. The converse also holds.

The mean curvature of V_m for the normal ${}_\tau\xi^i$ is defined by [8]

$${}^\tau\Omega = {}^\tau\Omega_{\alpha\beta}a^{\alpha\beta}$$

and is related to the mean curvature of V'_m for the normal ${}_\tau\xi'^i$ by

$$(5.8) \qquad {}^\tau\Omega' = e^{-\sigma}({}^\tau\Omega - \bar{\sigma}_k\,{}_\tau\xi^k).$$

The mean curvature normal of V_m is defined by

$$\bar{\xi}^i = {}^\tau\Omega_{\alpha\beta}a^{\alpha\beta}\,{}_\tau\xi^i$$

and is related to the mean curvature normal of V'_m by

$$(5.9) \qquad \bar{\xi}'^i = e^{-2\sigma}(\xi^i - {}_\tau\Sigma\bar{\sigma}_k\,{}_\tau\xi^k\,{}_\tau\xi^i).$$

The mean curvature of V_m is defined as the mean curvature of V_m for the mean curvature normal and is denoted by M. From (5.8), we have

$$(5.10) \qquad M' = e^{-\sigma}(M - \bar{\sigma}_k\xi^k).$$

From these relations we may state: *A necessary and sufficient condition that mean curvatures of V_m and V'_m be related by $M' = e^{-\sigma}M$, is that V_m contain a congruence of σ-curves.*

If equation (5.9) is treated as was (5.4), we find that mean curvature normals of V_m and V'_m correspond if the direction, $\bar{\sigma}_k\xi^k\xi^i - \bar{\sigma}^i$, lies in V_m or if σ^i coincide with the direction ξ^i at a point. From a relation similar to (5.5) obtained by replacing ζ^i by ξ^i, it is seen that the converse also holds. Hence: *The mean curvature normal to V_m (not containing σ-curves) corresponds to the mean curvature normal to V'_m if and only if the vector, $\bar{\sigma}_k\xi^k\xi^i - \bar{\sigma}^i$, lies in V_m or the direction σ^i coincide with the direction ξ^i.*

The principal normals of a curve in V_m and of the corresponding curve in V'_m are related by an equation similar to (2.4),

$$\nu'^i/\rho_g = e^{-2\sigma}(\nu^i/\rho_g + \tilde{\sigma}_\gamma\,{}_1\lambda^\gamma\,{}_1\lambda^i - \tilde{\sigma}^i),$$

where $\tilde{\sigma}^i$ are the components in V_n of the direction $\tilde{\sigma}^\alpha = a^{\alpha\beta}\tilde{\sigma}_\beta = a^{\alpha\beta}(\partial\sigma/\partial y^\beta)$ in V_m. If equation (2.6) be written for C' and primed quantities be replaced by unprimed, we obtain

[8] R. G., p. 168.

$$\bar{\sigma}^i = \tilde{\sigma}^i + \sum_{\tau=1}^{n-m} \bar{\sigma}_k \, {}_\tau\xi^k \, {}_\tau\xi^i, \tag{5.11}$$

which says that the curves defined by the congruence $\tilde{\sigma}^i$, called $\tilde{\sigma}$-curves, are projections of the σ-curves on the V_m. These $\tilde{\sigma}$-curves play the same rôle in the subspace that the σ-curves do in V_n.

From the last theorem of section three and from the fact that the given conformal transformation induces a transformation with constant coefficient of magnification on a subspace if and only if that subspace lie in the H_σ, it follows that: *A necessary and sufficient condition that a geodesic C in V_m correspond to a geodesic in V'_m is that C be a σ-curve or that V_m lie in H_σ.*

If the V_m contain a congruence of σ-curves, a geodesic in V_m will correspond to a geodesic in V'_m if and only if it be a $\tilde{\sigma}$-curve. It is to be noted that if the subspace contain no σ-curves, no geodesic in both V_n and V_m can correspond to a geodesic in V'_n and V'_m.

6. Properties of curves in corresponding hypersurfaces. Inasmuch as there is but one normal ξ^i to a V_{n-1} in a V_n, and this normal corresponds in direction to the normal to V'_{n-1} in V'_n, the results of the previous section are somewhat simplified in this case. Theorems concerning asymptotic lines and lines of curvature may be paraphrased directly.

In considering the correspondence of geodesics, it is seen that if the V_{n-1} contain σ-curves, the only geodesics which can correspond to geodesics are the σ-curves. If the V_{n-1} be normal to the σ-curves, i. e., H_σ, all geodesics correspond to geodesics. For a general hypersurface, a necessary and sufficient condition that a geodesic correspond to a geodesic is that C be a $\tilde{\sigma}$-curve. Since principal normals to a geodesic in V_{n-1} are normal to V_{n-1}, it follows that a geodesic in V_{n-1} corresponds to a geodesic in V'_{n-1} if and only if C and C' have corresponding principal normals. It can be shown further that if C is a $\tilde{\sigma}$-curve (i. e., $\nu' = e^{-\sigma}\nu^i$, $\rho'_g/\rho_g = e^\sigma$) and if C and C' have corresponding principal normals, from the relation (2. 6) written for C', by replacing the primed quantities in terms of the unprimed, then the principal normals of C are normal to the V_{n-1} and hence C and C' are geodesics in V_{n-1} and V'_{n-1}. Hence: *If a curve in a hypersurface V_{n-1} which does not contain a congruence of σ-curves and is not normal to the σ-curves, is a $\tilde{\sigma}$-curve, and if C and C' have corresponding principal normals, then C and C' are geodesic in V_{n-1} and V'_{n-1}, and conversely.*

7. Parallelism. From the relation (3. 4), it appears that corresponding directions are simultaneously parallel along C and C' (not σ-curves) if and only if

$$(7.1) \qquad \mu^i{}_{,j}\,{}_1\lambda^j = 0, \quad g_{ij}\sigma^i\mu^j = 0, \quad g_{ij}\,{}_1\lambda^i\mu^j = 0,$$

i. e., if and only if directions μ^i are parallel along C and are normal to the σ-surface of C.

If we write,

$$(7.2) \qquad \begin{aligned} {}_1\lambda^i &= l^a\,{}_a\sigma^i \\ \mu^i &= m^a\,{}_a\sigma^i \end{aligned} \qquad (\alpha = 1 \cdots n),$$

where ${}_a\sigma^i$ is the normal ennuple of the σ-curves, then the conditions for simultaneous parallelism become

$$(7.3a) \qquad {}_a\Sigma l^a m^a = 0,$$

$$(7.3b) \qquad m^1 = 0,$$

$$(7.3c) \qquad dm^a/ds = -\gamma^a{}_{\gamma\beta}m^\gamma l^\beta,$$

where s is the arc of C and the γ's are the Ricci coefficients of rotation for the ennuple ${}_a\sigma^i$.

Differentiating (7. 3a) and $\gamma_{a\beta}m^\gamma l^\beta = 0$ with respect to s, and making use of (7. 3c), we obtain the following system of $(n + 2)$ equations which the $2n$ quantities l^a, m^a must satisfy:

$$(7.4a) \qquad m^a(dl^a/ds) - \gamma^a{}_{\beta\gamma}l^a l^\gamma m^\beta = 0,$$

$$(7.4b) \qquad (dl^\beta/ds)\gamma^1{}_{\gamma\beta}m^\gamma - \gamma^1{}_{a\beta}\gamma^\gamma{}_{\delta\epsilon}m^\delta l^\beta l^\epsilon + (\partial\gamma^1{}_{\gamma\beta}/\partial x^i)\,m^\gamma l^\beta\,{}_a\sigma^i l^a = 0,$$

$$(7.4c) \qquad (dm^a/ds) + \gamma^a{}_{\gamma\beta}m^\gamma l^\beta = 0.$$

If equations (7. 4a) and (7. 4b) are dependent, we find that

$$(7.5) \qquad \gamma^1{}_{\gamma a} = 0 \qquad \gamma^1{}_{aa} = \rho$$

i. e., that ${}_1\sigma^i$ is a normal congruence, that ${}_2\sigma^i, \cdots, {}_n\sigma^i$ are canonical with respect to ${}_1\sigma^i$ and that lines of curvature of the hypersurfaces, H_σ, are completely indeterminate.[9] Conversely, if the hypersurfaces, H_σ, have completely indeterminate lines of curvature, the γ's for any orthogonal ennuple, and in particular for the normal ennuple of the σ-curves, satisfy the conditions

$$\gamma^1{}_{a\gamma} = 0, \qquad \gamma^1{}_{aa} = \gamma^1{}_{\beta\beta}$$

and a solution of the system (7. 4) will involve $(n - 1)$ instead of $(n - 2)$ arbitrary functions. Hence: *If the hypersurfaces, H_σ, have completely indeterminate lines of curvature, equations* (7. 4a) *and* (7. 4b) *are dependent and*

[9] R. G., p. 126

the general solution of equations (7.4) *involves* $(n-1)$ *arbitrary functions* $l^2, \cdots, l^n$; *otherwise a solution involves* $(n-2)$ *functions* $l^3, \cdots, l^n$. *In either case, the solution is uniquely determined by the arbitrary functions and a set of initial values satisfying the conditions,*

$$
{}_0m^1 = 0, \qquad \gamma^1{}_{\gamma\beta}\, {}_0m^\gamma\, {}_0l^\beta = 0,
$$
$$
{}_\alpha\Sigma\, {}_0m^\alpha\, {}_0m^\alpha = 1, \qquad {}_\alpha\Sigma\, {}_0m^\alpha\, {}_0l^\alpha = 0.
$$

Every such solution l^α, m^α *of equations* (7.4) *determines a curve on a* σ*-surface, along which directions* μ^i *are parallel, and such that corresponding directions are parallel along* C'.

We shall now consider the geometric properties of curves (other than σ-curves) which admit simultaneous parallelism. We have already seen that the directions parallel along these curves must be normal to the σ-surfaces of the curves. For the normal ${}_\gamma\xi^i$ to a V_2 immersed in a V_n we have the relation [10]

$$
(7.6) \qquad {}_\gamma\xi^i{}_{,j}\, {}_1\lambda^j = -{}^\gamma\Omega_{\alpha\beta} a^{\alpha\delta} x^i{}_{,\delta}\, {}_1\lambda^\beta + {}_\tau\Sigma \mu_{\tau\gamma/\beta}\, {}_\tau\xi^i\, {}_1\lambda^\beta \qquad \begin{array}{l}(\alpha, \beta, \delta = 1, 2)\\ (\tau, \gamma = 1 \cdots n-m)\end{array}
$$

where [11]

$$
\mu_{\tau\gamma/\beta}\, {}_1\lambda^\beta = g_{ij}\, {}_\gamma\xi^j{}_{,k}\, {}_1\lambda^k\, {}_\tau\xi^i.
$$

If directions ${}_\gamma\xi^i$ are parallel along C, (7.6) reduces to

$$
(7.7) \qquad {}^\gamma\Omega_{\alpha\beta} a^{\alpha\delta} x^i{}_{,\delta}\, {}_1\lambda^\beta = 0.
$$

Multiplying this by σ_i and then by ${}^1\lambda_i$ gives respectively

$$
(7.8) \qquad \begin{aligned} {}^\gamma\Omega_{\alpha\beta}\, {}_1\lambda^\alpha \sigma^\beta &= 0, \\ {}^\gamma\Omega_{\alpha\beta}\, {}_1\lambda^\alpha\, {}_1\lambda^\beta &= 0. \end{aligned}
$$

Since C is not a σ-curve, these equations will be satisfied only when C is a doubly counting asymptotic line for the normal ${}_\gamma\xi^i$. Hence we may state: *Necessary and sufficient conditions that directions,* ${}_\gamma\xi^i$, *along* C, *normal to its* σ*-surface, be parallel are:* 1) *that for the given normal,* C *be a doubly counting asymptotic line on its* σ*-surface, and* 2) *that the* $(n-3)$ *vectors* $\mu_{\tau\gamma/\beta}$ *determined by the given normal coincide with the directions of the normal to* C *in its* σ*-surface, i. e.,* $\mu_{\tau\gamma/\beta}\, {}_1\lambda^\beta = 0$, $(\tau = 1, \cdots, n-2;\ \tau \neq \gamma)$.

In three-space relation (7.6) reduces to

$$
(7.9) \qquad {}_\gamma\xi^i{}_{,j}\, {}_1\lambda^j = -{}^\gamma\Omega_{\alpha\beta} a^{\alpha\delta} x^i{}_{,\delta}\, {}_1\lambda^\beta.
$$

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[10] R. G., p. 168.

[11] R. G., p. 160.

Hence, if $n=3$, a necessary and sufficient condition that directions ${}_\gamma\xi^i$ normal to the σ-surface of C, be parallel along C, is that C be a doubly counting asymptotic line on its σ-surface.

If we differentiate covariantly the equations

$$g_{ij}\,{}_1\lambda^i\,{}_\gamma\xi^j=0, \qquad g_{ij}\sigma^i\,{}_\gamma\xi^j=0, \qquad (i,j=1\cdot\cdot\cdot n)$$

and assume that C is a doubly counting asymptotic line for the normal ${}_\gamma\xi^i$, i. e., that (7. 8) hold, it is found that ${}_2\lambda^i$ and $\sigma^i{}_{,k}\,{}_1\lambda^k$ lie in the σ-surface of C. If $n=3$, and if ${}_2\lambda^i$ and $\sigma^i{}_{,k}\,{}_1\lambda^k$ lie in the σ-surface of C, it follows conversely that C is a doubly counting asymptotic line on its σ-surface. Accordingly: in V_3, a necessary and sufficient condition that a curve C (not geodesic in V_n), asymptotic on its σ-surface, be doubly counting asymptotic, is that the associate directions with respect to C of the σ-curves at points of C be directions in the σ-surface of C.

If C is geodesic in V_3 and if σ^i be parallel along C, C is a doubly counting asymptotic line on its σ-surface. Therefore: *In V_3, if the σ-curves admit among their transversals a geodesic, that geodesic is a doubly counting asymptotic line on its σ-surface.*

From these results we conclude that the only curves in V_3 which admit simultaneous parallelism are σ-curves, geodesics along which the σ-curves are parallel, and curves asymptotic on their σ-surfaces such that the associate directions of the σ-curves with respect to the C curves are directions in the σ-surfaces.

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SURFACES WHOSE ASYMPTOTIC CURVES ARE TWISTED CUBICS.*

By E. P. Lane and M. L. MacQueen.

1. Introduction. The purpose of this paper is to put on record some results relative to the problem of determining all analytic non-ruled surfaces whose asymptotic curves are twisted cubics. The problem is reduced to the integration of an ordinary differential equation. Some special cases are considered, in which interesting results can be deduced from this equation. Some examples of surfaces whose asymptotic curves are twisted cubics are discussed, and reference is made to Terracini's work on this subject.

2. Analytic Basis. This section summarizes portions of the classical analytical theory of the projective differential geometry of curves and surfaces which are used in later developments. In ordinary space, in which a point has projective homogeneous coördinates $x^{(1)}, \cdots, x^{(4)}$, the parametric vector equation of an analytic non-ruled surface is

$$x = x(u, v), \tag{1}$$

the parameters being u, v. If the asymptotic curves on the surface are the parametric curves, the coördinates x satisfy a system of two partial differential equations which can be reduced to the form

$$\begin{aligned} x_{uu} &= px + \theta_u x_u + \beta x_v, \\ x_{vv} &= qx + \gamma x_u \;\, + \theta_v x_v \end{aligned} \qquad (\theta = \log \beta\gamma), \tag{2}$$

subscripts indicating partial differentiation, and the coefficients being functions of u, v, which satisfy certain integrability conditions.

The parametric vector equation of an analytic curve is

$$x = x(t), \tag{3}$$

the parameter being t. These coördinates x satisfy an ordinary differential equation of the form

$$x^{iv} + 4p_1 x''' + 6p_2 x'' + 4p_3 x' + p_4 x = 0 \qquad (x' = dx/dt, \cdots), \tag{4}$$

the coefficients being functions of t. Let P_2, P_3, P_4 be defined by the formulas

* Received October 25, 1937.

$$
\begin{aligned}
&P_2 = p_2 - p_1^2 - p_1' \\
(5)\qquad &P_3 = p_3 - 3p_1p_2 + 2p_1^3 - p_1'', \\
&P_4 = p_4 - 4p_1p_3 + 6p_1^2p_2 - 6p_1'p_2 - 3p_1^4 + 6p_1^2p_1' + 3p_1'^2 - p_1'''.
\end{aligned}
$$

Then two invariants θ_3, θ_4 of the differential equation are defined by the formulas

$$
(6)\qquad
\begin{aligned}
\theta_3 &= P_3 - \tfrac{3}{2}P', \\
\theta_4 &= P_4 - \tfrac{81}{25}P_2^2 - 2P_3' + \tfrac{6}{5}P_2''.
\end{aligned}
$$

It is well known that the integral curves of equation (4) belong to linear complexes in case $\theta_3 = 0$, and are twisted cubics in case $\theta_3 = \theta_4 = 0$. Twisted cubics thus appear as a subclass of the class of all curves belonging to linear complexes.

3. Surfaces whose asymptotic curves belong to linear complexes. The problem of determining all non-ruled surfaces whose asymptotic curves belong to linear complexes has been completely solved. As our method of attack on the problem before us consists in selecting from these surfaces the class of surfaces whose asymptotic curves are twisted cubics, it will be useful to state here some known results relative to surfaces whose asymptotic curves belong to linear complexes.

It is known that, in case the integral surfaces of equations (2) have the property that their asymptotic curves belong to linear complexes, the coefficients β, γ in the equations can be specialized so that

$$
(7)\qquad \beta = \gamma = \frac{\sqrt{U'V'}}{U+V} \qquad (U'V' \neq 0),
$$

where U is an arbitrary function of u alone, and V of v alone. Moreover, the coefficients p, q are given by the formulas

$$
(8)\qquad
\begin{aligned}
2p &= 3l_{uu} - \tfrac{3}{2}l_u^2 - 3\beta l_v - U_1, \\
2q &= 3l_{vv} - \tfrac{3}{2}l_v^2 - 3\beta l_u - V_1,
\end{aligned}
$$

wherein l is defined by

$$
(9)\qquad l = \log \beta
$$

and U_1, V_1 by

$$
(10)\qquad U_1 = \frac{DU^2 + EU + F}{U'}, \qquad V_1 = \frac{DV^2 - EV + F}{V'},
$$

where D, E, F are arbitrary constants.

4. Conditions for twisted cubics. Analytic conditions necessary and sufficient that the asymptotic curves on a surface not only belong to linear

complexes but actually are twisted cubics can now be computed. If the coefficients of equations (2) satisfy the conditions of Section 3, then the coefficients of the equation of the form (4) for the asymptotic u-curves can be calculated, and are found to be given by the formulas

$$
\begin{aligned}
2p_1 &= -3l_u, \\
6p_2 &= 11l_u^2 - 5l_{uu} - 2p - 3\beta l_v, \\
4p_3 &= -2p_u + 6pl_u + 10l_ul_{uu} - 2l_{uuu} - 3\beta^3 - 6l_u^3 + 6\beta l_ul_v, \\
p_4 &= -p_{uu} + 4p_ul_u + pl_{uu} - \beta p_v - \beta^2 q - 3pl_u^2 + p^2 + 3\beta pl_v.
\end{aligned}
\tag{11}
$$

Then the functions P_2, P_3, P_4 defined by the formulas (5) are found to be given by

$$
\begin{aligned}
6P_2 &= \tfrac{1}{2}S + U_1, \\
4P_3 &= \tfrac{1}{2}S' + U_1', \\
4P_4 &= U_1^2 + 3U_1'' \qquad (U_1' = dU_1/du, \cdots),
\end{aligned}
\tag{12}
$$

where S is the Schwarzian derivative of U with respect to u, defined by the formula

$$S = \left(\frac{U''}{U'}\right)' - \tfrac{1}{2}\left(\frac{U''}{U'}\right)^2. \tag{13}$$

The invariants θ_3, θ_4 defined by the formulas (6) can now be calculated for the u-curves. Of course we find $\theta_3 = 0$. But calculating θ_4 and setting the result equal to zero, we obtain the following *necessary and sufficient condition that the asymptotic u-curves be twisted cubics:*

$$S'' + \tfrac{3}{20}S^2 + \tfrac{3}{5}SU_1 - \tfrac{16}{15}U_1^2 - 3U_1'' = 0. \tag{14}$$

The analogous condition that the asymptotic v-curves be twisted cubics is

$$T'' + \tfrac{3}{20}T^2 + \tfrac{3}{5}TV_1 - \tfrac{16}{15}V_1^2 - 3V_1'' = 0, \tag{15}$$

where T is defined by

$$T = \left(\frac{V''}{V'}\right)' - \tfrac{1}{2}\left(\frac{V'}{V''}\right)^2. \tag{16}$$

Because of the analogy between equations (14) and (15) it will be sufficient to confine our discussion to equation (14). This is to be regarded as an ordinary differential equation of the fifth order for the determination of U as a function of u, and the problem of determining all non-ruled surfaces whose asymptotic curves are twisted cubics is thus in effect reduced to the solution of this differential equation.

5. Analytical theorems. A well-known theorem states that the general

solution of the third-order differential equation $S=0$ is obtained by setting U equal to a linear fractional function of u with constant coefficients. In the course of our investigation, we have been led to some theorems of a kindred nature, which we have not found in the literature, and which we shall state here. The first three theorems relate to the Schwarzian derivative S defined by the formula (13), and their proofs, being immediate, will be omitted.

THEOREM 1. *The general solution of the differential equation*

$$S=k \qquad (k=\text{const.}\neq 0)$$

is obtained by integrating the differential equation

$$U'=PU^2+QU+R,$$

where P, Q, R are constants such that

$$Q^2-4PR=-2k.$$

THEOREM 2. *An integral of the differential equations*

$$S=\frac{AU^2+BU+C}{U'}=k \qquad (k=\text{const.}\neq 0)$$

is

$$U'=\frac{AU^2+BU+C}{k}$$

where A, B, C are constants such that $B^2-4AC=-2k^3$.

THEOREM 3. *An integral of the differential equation*

$$S=\frac{AU^2+BU+C}{U'}\neq\text{const.},$$

where A, B, C are arbitrary constants, is

$$(17) \quad \frac{AU^2+BU+C}{U'}\left(\frac{U''}{U'}\right)^2-2(2AU+B)\frac{U''}{U'} + 4AU'+\left(\frac{AU^2+BU+C}{U'}\right)^2=c'$$

where c' is an arbitrary constant.

It is also easy to verify the following statement:

THEOREM 4. *The Schwarzian derivative S defined by the formula* (13), *and the function U_1 defined by the first of the formulas* (10), *satisfy the equation*

$$(18) \qquad U_1''' + 2SU_1' + S'U_1 = 0,$$

and also satisfy the equation

$$(19) \qquad U_1'^2 - 2U_1U_1'' - 2U_1^2S = E^2 - 4DF.$$

6. Special cases. In certain special cases interesting conclusions can be drawn from equation (14), either alone or in the presence of the equations of Section 5. In particular, the following four theorems can be proved; the details of the demonstrations are so elementary as not to need to be reproduced here.

THEOREM 5. *If* $S = 0$, then $U_1 = 0$.

THEOREM 6. *If* $U_1 = 0$, *then* $S'^2 + \frac{1}{10}S^3 = \text{const.}$

THEOREM 7. *If* $S = k = \text{const.} \neq 0$, *then either* $U_1 = 3k/4$ *or* $U_1 = -3k/16$. *In the first alternative we have*

$$E^2 - 4DF = -\tfrac{9}{8}k^3, \qquad U' = \frac{DU^2 + EU + F}{3k/4},$$

and in the second,

$$E^2 - 4DF = -\tfrac{9}{128}k^3, \qquad U' = \frac{DU^2 + EU + F}{-3k/16}.$$

THEOREM 8. *If* $U_1 = \text{const.} \neq 0$, *then* $S = \text{const.} \neq 0$.

In the special case in which

$$S = T = U_1 = V_1 = 0,$$

we have $D = E = F = 0$, and U, V can be expressed in the form

$$U = \frac{au + b}{cu + d}, \quad V = \frac{a'v + b'}{c'v + d'} \qquad (ad - bc \neq 0, a'd' - b'c' \neq 0),$$

in which $a, b, c, d, a', b', c', d'$ are arbitrary constants. Choosing a proportionality factor for these constants so that

$$ad - bc = 1, \qquad a'd' - b'c' = 1,$$

we are able to integrate equations (2) completely in this case and thus to prove the following theorem.

THEOREM 9. *The asymptotic curves are twisted cubics, and the directrix curves are indeterminate, on the surface whose parametric equations referred to its asymptotics are*

$$
\begin{aligned}
x_1 &= 1,\\
x_2 &= (\tfrac{1}{2}cu^2 + du) - (\tfrac{1}{2}c'v^2 + d'v),\\
x_3 &= (\tfrac{1}{2}au^2 + bu) + (\tfrac{1}{2}a'v^2 + b'v),\\
x_4 &= \tfrac{1}{6}(u^3 + v^3) + (\tfrac{1}{2}au^2 + bu)(\tfrac{1}{2}c'v^2 + d'v)\\
&\qquad + (\tfrac{1}{2}cu^2 + du)(\tfrac{1}{2}a'v^2 + b'v),
\end{aligned}
$$

and on every surface projectively equivalent to this one. These surfaces are algebraic and of order six or eight.

Another interesting special case is that characterized by the conditions

$$U_1 \neq \text{const.}, \quad S \neq \text{const.}, \quad U_1 = rS, \quad r = \text{const.} \neq 0. \tag{20}$$

In this case equation (18) gives at once

$$S''' + 3SS' = 0,$$

and integration then leads to

$$2S'' + 3S^2 = c', \tag{21}$$

where c' is an arbitrary constant. Substitution of our expressions for U_1 and S'' in equation (14) leads to an identity in S, from which we conclude that $c' = 0$ and that *the ratio r can have only one or the other of the two values $r = 9/2$ or $r = 9/32$.* Equation (19) yields

$$S'^2 + S^3 = \frac{E^2 - 4DF}{r^2}.$$

If we place

$$A = D/r, \qquad B = E/r, \qquad C = F/r,$$

and refer to Theorem 3 we see that equation (17) with $c' = 0$ is valid.

7. Examples. Three examples of surfaces whose asymptotic curves are twisted cubics and which have been considered by different geometers, will now be adduced. It happens that these examples belong to special cases mentioned above.

The asymptotic curves on the minimal surface of Enneper are known[1] to be twisted cubics. Parametric equations of this algebraic surface of order nine, referred to its asymptotic curves, are

$$
\begin{aligned}
x_1 &= 1,\\
x_2 &= (u + v)[3 + 3(u - v)^2 - (u + v)^2],\\
x_3 &= (u - v)[3 + 3(u + v)^2 - (u - v)^2],\\
x_4 &= 12uv.
\end{aligned} \tag{22}
$$

[1] Darboux, *Leçons sur la théorie générale des surfaces*, second edition, vol. 1 (1914), pp. 374-376.

Application of our general theory to this surface offers no difficulty. Omitting the calculations, we shall merely state that this surface belongs to the special case characterized by the conditions (20) with $r = 9/32$.

Wilczynski remarked [2] that the surface whose parametric equations, referred to its asymptotics, are

$$\begin{aligned} &x_1 = 1, \qquad x_2 = u + v, \qquad x_3 = U' - V', \\ &x_4 = -2(U' - V')(u - v) + 4(U + V), \end{aligned} \tag{23}$$

where U and V are cubic polynomials in u alone and v alone, respectively, is an algebraic surface of the sixth order on which the directrix curves are indeterminate and on which the asymptotic curves are twisted cubics. Application of our general theory to this surface of Wilczynski shows that it belongs to the special case characterized by the conditions $S = U_1 = 0$, so that U is a linear fractional function of u, and $D = E = F = 0$.

Enriques has studied [3] the surface whose algebraic equation is

$$(x_1x_3 - x_2^2)^3 k = (x_1^2 x_4 + 2x_2^3 - 3x_1x_2x_3)^2 \qquad (k = \text{const.}), \tag{24}$$

which has also been investigated [4] recently by Emma Castelnuovo. As Enriques pointed out, this surface has not only the property that it admits a two-parameter family of projective transformations into itself, but also the property that its asymptotic curves are twisted cubics. Parametric equations of this surface of Enriques, referred to its asymptotic curves, are

$$\begin{aligned} &x_1 = 1, \quad x_2 = \frac{1+h}{2}u + \frac{1-h}{2}v, \quad x_3 = \frac{1+h}{2}u^2 + \frac{1-h}{2}v^2, \\ &x_4 = \frac{(1+h)^2}{4h}u^3 - 3\,\frac{1-h^2}{4h}u^2v + 3\,\frac{1-h^2}{4h}uv^2 - \frac{(1-h)^2}{4h}v^3, \end{aligned} \tag{25}$$

where

$$h^2 = \frac{4}{4+k}.$$

Application of our general theory shows that this surface is a special surface

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[2] Wilczynski, Abstract, *Bulletin of the American Mathematical Society*, vol. 20 (1913-14), p. 312.

[3] Enriques, "Le superficie con infinite trasformazioni proiettive in sè stesse," *Atti del R. Istituto Veneto di scienze, lettere ed arti*, ser. 7, vol. 4 (1893), pp. 1590-1635. See also Enriques, Intorno alla Memoria "Le superficie con infinite trasformazioni proiettive in sè stesse," *ibid.*, vol. 5 (1894), and Lie, "Bestimmung aller Flächen, die eine continuerliche Schaar von projectiven Transformationen gestatten," *Berichte der Gesellschaft der Wissenschaften zu Leipzig*, vol. 47 (1895).

[4] E. Castelnuovo, "Di una classe di superficie razionali che ammettono ∞^2 trasformazioni proiettive in sè," *Rendiconti dei Lincei*, ser. 6, vol. 24 (1936), pp. 342-346.

of Wilczynski. In fact, the function U of the general theory is linear in u for the surfaces of Enriques, whereas U is a linear fractional function of u for the general surface of Wilczynski.

8. Terracini's formulas. Terracini has shown [5] that if a non-ruled surface has the property that its asymptotic curves belong to linear complexes, then parametric equations of the surface, referred to its asymptotics, can be written in one or another of the following three forms, according to the nature of a certain quadric, commonly called the quadric of Sullivan, associated with the surface:

$$\begin{aligned}
&x_1 = (V' - U')u + 2U, && x_2 = (V' - U')v - 2V,\\
&x_3 = (V' - U')uv - 2(Vu - Uv), && x_4 = V' - U';\\[1ex]
(26)\quad &x_1 = (U' - V')u - 2U, && x_2 = (U' - V')v + 2V,\\
&x_3 = 2uv, && x_4 = -2;\\[1ex]
&x_1 = (U' - V')(u - v) - 2(U + V), && x_2 = U' - V',\\
&x_3 = u + v, && x_4 = 1,
\end{aligned}$$

where U is an arbitrary function of u alone, and V of v alone. In all three cases the coefficients of the differential equation of the form (4) for the u-curves are found by actual calculation to be given by the formulas

$$2p_1 = -R, \qquad 6p_2 = -R' + R^2, \qquad p_3 = p_4 = 0, \tag{27}$$

where R is defined by placing

$$R = \frac{U^{iv}}{U'''}. \tag{28}$$

Calculating the invariant θ_4 for this equation and setting this invariant equal to zero, we obtain the following differential equation:

$$10R''' - 30RR'' + 32R^2R' - 19R'^2 - 4R^4 = 0. \tag{29}$$

There is of course a similar equation for the v-curves. Thus the problem of determining parametric equations of all non-ruled surfaces whose asymptotic curves are twisted cubics is reduced to the problem of integrating the differential equation (29) of the third order to obtain the function R, and then performing four quadratures on equation (28) to obtain the function U.

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[5] Terracini, "Sulle superficie le cui asintotiche dei due sistemi sono cubiche sghembe," *Atti della Società dei Naturalisti e Matematici di Modena*, ser. 5, vol. 5 (1919-20).

ON HYPERGROUPS, MULTIGROUPS, AND PRODUCT SYSTEMS.*

By L. W. Griffiths.

1. Introduction. In this paper there is considered an abstract system whose elements are classes. Each class is an unordered set of marks, selected from a fundamental set Σ of distinct marks. The marks in a class are distinct if and only if this property is specifically stated for the system. The classes are distinct.

A product system is a system satisfying two postulates. First, the system is closed with respect to an addition process which is associative and commutative. Second, the system is closed with respect to a multiplication process which is associative, and which is distributive, on the right and on the left, with respect to the addition process. A division system Δ is a product system which satisfies postulate III or postulate III′. Thus, in particular, if a product system contains the subset Ω of all classes each of which consists of exactly one mark, then for every pair, A and B, of classes in Ω there are two classes, B' and B'', each in Ω, such that BB' contains A and $B''B$ contains A.

A Γ system is formed from a division system which contains Ω by replacing each class in Ω by the mark of which that class consists. It is proved that the fundamental set Σ in a Γ system is a hypergroup as defined by Marty.[1] Conversely, a Marty hypergroup is embedded as Σ in the Γ system formed from its product system. If it is postulated that a Γ system has a mark in Σ whose properties are suggested by those of the identity in group theory, then Σ is a regular hypergroup. If it is postulated that this mark is unique, and that for each mark in Σ there is in Σ a mark whose properties are suggested by those of inverse elements in group theory, then Σ is a completely regular hypergroup. A further condition makes Σ a normal hypergroup of Marty.[2] Conversely, each of these types of hypergroups is embedded in the Γ system of its products, and this Γ system has the corresponding property stated above.

It is proved that a multigroup as defined by Ore [3] is a completely regular

* Received November 17, 1937; Revised slightly, December 20, 1937.

[1] F. Marty, "Sur une généralisation de la notion de groupe," *Särtryck ur Förhandlinger vid Åttonde Skandinaviska Matematikerkongressen i Stockholm* 1934, pp. 45-49.

[2] F. Marty, "Rôle de la notion d'hypergroupe dans l'étude des groupes non abeliens," *Comptes Rendus de l'Academie des Sciences*, vol. 201 (1935), pp. 636-638.

[3] O. Ore, "Structures and group theory I," *Duke Mathematical Journal*, vol. 3 (1937), pp. 149-174.

hypergroup of Marty for which multiplication by the identity on one side is unique.

If a Γ system satisfies the preceding conditions for a Marty completely regular hypergroup except that there may be more than one identity element in Σ, and if there exists a positive integer n such that the product of every pair of marks in Σ is a class consisting of precisely n marks, then Σ is a hypergroup as defined by Wall.[4] Conversely, a Wall hypergroup is a special kind of Marty hypergroup, namely one for which such an integer n exists and for which an identity and inverse elements exist, and hence is embedded in a Γ system as is a Marty hypergroup.

A division system which contains Ω particularizes to a group if it is merely Ω and if in postulate III B' and B'' are unique. Then the embedded hypergroup is the group. A division system which does not contain Ω does not particularize to a group whose elements are marks in Σ. However, it is proved in Theorem 7 that if the classes of such a division system are regarded as marks of a new fundamental system Σ', then the product system of Σ' is Σ'. Hence either a division system contains a Marty hypergroup Ω or this division system as Σ' is a Marty hypergroup in which multiplication is unique. Thus either a division system particularizes to a group whose elements are the marks in Σ or it particularizes to a group whose elements are the marks in Σ'.

In section 7 it is proved that the direct product of two finite groups is simply isomorphic with a subset of classes in a division system which does not contain Ω. However, as stated above, this subset of classes is a subgroup of product system Σ'.

A division system is a generalization of the abstract system group, since it has the closure property for multiplication, since either division is postulated or the existence of elements analogous to the identity and inverse elements of group theory is postulated, and since it particularizes to a group in that one of the two senses explained above which corresponds to that one of postulates III or III′ used in its definition. Furthermore, as proved in section 3, if further conditions are to be placed on a product system, of a type suggested by the usual postulates for a group, a division system is the most general type so obtained.

There is defined a Σ algebra, and therefore in particular a hypergroup algebra, which is analogous to the ordinary group algebra.

2. Postulational definition of a Δ system and of a Γ system. Consider a set Σ of distinct marks, $a, b, c, \cdots$. If k is a positive integer, then $a_1, \cdots, a_k$

[4] H. S. Wall, "Hypergroups," *American Journal of Mathematics*, vol. 59 (1937), pp. 77-98.

is a notation for a finite set of k of these marks, and $a_1, a_2, \cdots$ is a notation for an infinite set. In these sets the marks are not necessarily distinct. R, S, T, M, and other capital letters near the end of the alphabet represent classes, $\{a_1, \cdots, a_k\}$ and $\{a_1, a_2, \cdots\}$. The marks in a class are not ordered, nor are they necessarily distinct. A, B, C, A_1, and other capital letters near the beginning of the alphabet, with or without subscripts, represent classes each of which consists of exactly one mark; thus, for example, A is the class $\{a\}$, A_1 the class $\{a_1\}$. Ω is the set of all such classes A. Thus Ω is in one to one correspondence with Σ. The class R may be a class in Ω, in particular. A system of classes may or may not contain classes in Ω.

Two classes are equal if and only if the set of marks in the one class is the set of marks in the other. The class R includes the class S, in notation $R \supset S$, means that the set of marks in S is a subset of the set of marks in R. $R \supset S$ does not exclude $R = S$.

An addition of classes which is illustrated in sections 4 and 5 of this paper is the following. The system of classes allows no repetition of marks in a class. The sum of two classes is the class whose set of marks is the set of distinct marks among the set of all the marks of the first class and all those of the second class. An addition of classes for a system in which repetition of marks in a class is allowed is illustrated in section 6. It is the following: the sum of two classes is the class whose set of marks is the set of all the marks of the first class and all those of the second class. Each of these addition processes is associative and commutative.

A product system is a system of distinct classes which satisfies the first two of the following postulates.

Postulate I. For every pair, R and S, of classes in the system there is in the system a unique class $R + S$. This addition process is associative and commutative.

Postulate II. For every pair, R and S, of classes in the system there is in the system a unique class RS. This multiplication process is associative, and it is distributive, on the right and on the left, with respect to the addition process of postulate I.

Postulate III. The system contains Ω, and for every pair, A and B, of classes in Ω there are two classes, B' and B'', each in Ω, such that $BB' \supset A$ and $B''B \supset A$. B' is not necessarily unique, nor is B''.

Postulate III'. The system does not contain Ω (although it may contain one or more classes in Ω), and for every pair, R and S, of classes in the system

there are two classes S' and S'', each in the system, such that $SS' \supset R$ and $S''S \supset R$. S' is not necessarily unique, nor is S''.

A division system Δ is a product system which satisfies postulates I, II, and III, or postulates I, II, and III′. A Γ system is formed from a Δ system which contains Ω in the following manner. From the definition of A in Ω a one to one correspondence of Ω to Σ is established by $A \leftrightarrow a$ if and only if $A = \{a\}$. A Γ system is formed from a Δ system which contains Ω by replacing each class A in Ω by the corresponding mark in Σ. For a Γ system addition and multiplication are defined, and postulates I, II, and III stated, by making these replacements in the definitions and postulates for the Δ system. A Γ system is simply isomorphic to the Δ system from which it is derived, and the correspondence is preserved under multiplication and addition. To Ω in the Δ system corresponds Σ in the Γ system. A corresponding Δ system and Γ system have precisely the same properties.

3. Related systems. Certain systems are considered having classes whose properties are suggested by those of the identity and inverse elements in group theory. They are related to Δ systems in the following theorems.

THEOREM 1. *If a system of distinct classes contains Ω, and satisfies postulates I and II, and the following postulates IV and V, then it satisfies postulate III.*

Postulate IV. There exists a class E in Ω such that for every A in Ω it is true that $AE \supset A$ and $EA \supset A$. (E is not necessarily unique.)

Postulate V. There exists a class E satisfying postulate IV such that for every A in Ω there exist A_1 and A_2 in Ω such that $AA_1 \supset E$ and $A_2A \supset E$. (A_1 is not necessarily unique, nor is A_2).

To prove Theorem 1, let A and B be arbitrary classes in Ω. Then by the associative and distributive laws applied to the equations corresponding to the inclusions it is seen that $BB_1A \supset EA \supset A$. Therefore there is a class B', such that $B' \subset B_1A$ and B' is in Ω, and that $BB' \supset A$. Similarly there is a class B'' such that B'' is in Ω and $B''B \supset A$. Hence postulate III is satisfied.

THEOREM 2. *If a system of distinct classes does not contain Ω, and if it satisfies postulates I and II and the following postulates VI and VII, then it satisfies postulate III′.*

Postulate VI. There exists a class M in the system such that for every

class R in the system it is true that $RM \supset R$ and $MR \supset R$. (M is not necessarily unique.)

Postulate VII. There exists a class M satisfying postulate VI such that for every R in the system there exist two classes R_1 and R_2, each in the system, such that $RR_1 \supset M$ and $R_2R \supset M$. (R_1 is not necessarily unique, nor is R_2.)

To prove Theorem 2, let R and S be arbitrary classes in the system. Then if S_1 and S_2 are determined in accordance with postulate VII, by the associative and distributive laws and postulate VI, it is true that S_1R and RS_2 have the same properties as S' and S'' respectively of postulate III′.

THEOREM 3. *If a system of distinct classes contains* Ω, *and satisfies postulates I, II, VI and VII, then it satisfies the following postulate* III_0, *and then it satisfies postulate III.*

Postulate III_0. For every pair, R and S, of classes in the system there are two classes, S' and S'', each in the system, such that $SS' \supset R$ and $S''S \supset R$. (S' is not necessarily unique, nor is S''.)

To prove Theorem 3, let A and B be arbitrary classes in Ω. Then III_0 holds as in the proof of Theorem 2. Then by postulate III_0 there exists a class X in the system such that $BX \supset A$. Then by the distributive law there is a class B' in Ω such that $BB' \supset A$. Similarly there is a class B'' in Ω such that $B''B \supset A$.

THEOREM 4. *If a system of classes satisfies postulates II and III, and the following postulate I′, then it satisfies* III_0.

Postulate I′. The system is closed with respect to an addition process which is associative and commutative, and if it contains any class containing infinitely many marks then it contains all classes containing infinitely many marks.

To prove Theorem 4, let R and S be arbitrary classes in the system. If $R = A_1 + \cdots + A_k$, and $S = B_1 + \cdots + B_j$, and if A_r and B_s are arbitrary summands in R and S respectively, then there exist by postulate III classes C_{rs} in Ω such that $B_sC_{rs} \supset A_r$. Hence the class whose summands are precisely all these classes C_{rs} is effective as S' in postulate III_0. Similarly there exists a class S'', if both R and S are finite classes. If either or both of R and S are infinite classes, postulate I is insufficient to guarantee such a sum of classes C in the system, but the last part of postulate I′ does insure the sum in the system.

THEOREM 5. *If a system of distinct classes satisfies postulates I′, II, IV and V and contains Ω, then it satisfies postulates VI and VII.*

The proof of Theorem 5 is similar to the proofs of the preceding theorems. Postulate I was insufficient for infinite classes R, while postulate I′ was sufficient.

Therefore, if a system contains Ω and further conditions are to be placed on it, of a type suggested by the usual postulates for a group, a more general system is obtained under postulates I, II, and III than under various combinations from I or I′, II, III_0, or IV and V, or VI and VII.

The following theorem is useful later in relating the properties of these systems to those of a Marty hypergroup, to those of a Wall hypergroup, and to those of a multigroup.

THEOREM 6. *If a system of distinct classes contains Ω and if the addition process satisfying postulate I is either of the two addition processes defined preceding postulate I, then a multiplication process satisfies postulate II if and only if it satisfies the following postulate II′.*

Postulate II′. For every A and B in Ω there is a unique class AB in the system. If S and T are two classes in the system, one of which at least is not in Ω, then ST is the sum of all products AB as A ranges over all summands A in S and B ranges over all summands B in T. For every A, B, and C in Ω it is true that $(AB)C = A(BC)$.

To prove Theorem 6, let R and S be arbitrary classes in the system. Then by postulate II there is a unique class RS in the system. Hence, in particular, the first statement in postulate II′ is true. The proof that the second statement in postulate II′ is true should be read first under the definition of addition for the case that no repetition of summands is allowed, and then under the definition of addition for the case that repetition is allowed. Let $S = A_1 + \cdots$ and $T = B_1 + \cdot\cdot\cdot$ be two arbitrary classes, at least one of which, for example T, is not in Ω. Then by the distributive law $ST = SB_1 + \cdot\cdot\cdot$. If S is in Ω then this is precisely the second statement in postulate II′. If S is not in Ω, then again by the distributive law $ST = (A_1B_1 + \cdots) + (A_2B_1 + \cdots) + \cdots$; this is precisely the second statement in postulate II′. The converse will be proved before any consideration of associativity of multiplication is taken. Let S, T, and U have summands A_i, B_j and C_k respectively. Then by the first two statements in postulate II′ it is true that $ST = \Sigma A_iB_j$ summed for all pairs with A_i in S and B_j in T; $SU = \Sigma A_iC_k$ summed for all pairs A_i in S and C_k in U; $S(T + U) = \Sigma A_iD_k$ summed for all pairs A_i in S and D_k in $T + U$. Then clearly $ST + SU = S(T + U)$.

To prove that postulate II implies the third statement in postulate II′ it is to be noted that, since multiplication is associative for all classes under postulate II, it is so in particular for all classes in Ω. Conversely, if postulate II′ holds then it is proved as follows that multiplication for all classes is associative. For, by the preceding, multiplication of classes is distributive with respect to addition. Then, with the notations already introduced for S, T, and U, it is true that $(ST)U = [\Sigma(A_iB_j)]U = \Sigma[(A_iB_j)C_k]$ where the first summation is with respect to all pairs A_i in S and B_j in T and the second summation is with respect to all triples A_i in S, B_j in T, and C_k in U. Similarly $S(TU) = \Sigma[A_i(B_jC_k)]$ where the summation is with respect to these same triples. Since the class $A_i(B_jC_k)$ is the class $(A_iB_j)C_k$ by postulate II′, it follows that $(ST)U = S(TU)$. This completes the proof of Theorem 6.

4. Relation of a Γ system to a Marty hypergroup. Consider a Γ system derived from a Δ system satisfying postulates I, II, and III and allowing no repetition of marks in a class. Then Σ is a hypergroup as defined by Marty. For the letters $A, B, C, \cdots$ in Marty's notation mean the marks $a, b, c, \cdots$ in this paper. He explicitly states postulate III, and the first and last sentences of postulate II′. No explicit definition is given of the expression $(AB)C$, for example, nor is it explained why from $AE \supset A$ it follows that $AEB \supset AB$. But an analysis of the proofs indicates that the matter can be treated by introducing classes and class addition, and by defining the product of two classes as in the second statement of postulate II′. No explicit statement is made by Marty that no repetition of marks is allowed in products. However this is implicit, for the finite case at least, in the proof that $\sum_{i=1}^{i=n} \alpha_i{}^A = n$, since it is not permitted that i be greater than n there. Hence, by Theorem 6, the Σ of a Γ system derived as stated in the first statement of this section is a hypergroup, and conversely a Marty hypergroup is embedded in the Γ system of its own product classes.

It follows immediately that, if a system satisfies the hypotheses of Theorem 1 with E unique and $A_1 = A_2$, the set Σ in the derived Γ system is a Marty completely regular hypergroup; and that a completely regular hypergroup is embedded in such a system of its product classes. If $EA = A$ and $AE = A$, then Σ is a Marty normal hypergroup.

5. Relation of a Γ system to an Ore multigroup. The letters $B_1, B_2, \cdots$ in Ore's notation mean the marks $b_1, b_2, \cdots$ in this paper. Repetition of marks in a product is not allowed, since the marks in a product constitute a subset of the elements of the multigroup, that is, of Σ. In the example of a multi-

group given by the co-sets of a group with respect to an arbitrary subgroup addition is used, but addition is not explicitly used in the general definition of a multigroup. Postulate II′ is explicitly stated, with the word "sum" replaced by "subset." Postulate IV is stated, with E unique and one of the inclusions an equality; postulate V is stated, with $A_1 = A_2$. Hence by Theorems 6 and 1 an Ore multigroup is a completely regular hypergroup of Marty for which multiplication by the identity on one side is unique. Hence if a Γ system satisfies postulates IV and V with these extra conditions on E and A_1, then Σ is a multigroup, and, conversely, a multigroup is embedded in such a Γ system of its products.

6. Relation of a Γ system to a Wall hypergroup. The letters $a, b, c, \cdots$ in Wall's notation mean the marks $a, b, c, \cdots$ in this paper. Repetition of marks is allowed in products. Addition is explicitly used, although no formal postulate is stated. Postulates II′, IV, and V with $A_1 = A_2$ are stated. There is introduced a positive integer n such that, in my notation, every product AB is a sum of exactly n classes in Ω. The bracket product is the class of all distinct marks in the ordinary product. Hence a Wall hypergroup of dimension n with bracket product multiplication is a Marty hypergroup with the condition of dimensionality imposed, and as such has the relation to a Γ system stated in section 4. A Wall hypergroup of dimension n with ordinary product multiplication is embedded as a set Σ in a Γ system of its own product classes. An nG_1 is a Wall hypergroup for which E is unique, and for which $A_1 = A_2$ and is unique for every A. The scalars, if existing, are a subset in a Wall hypergroup, and hence a subset in the Σ of the Γ system of the product classes.

7. The Γ system of a Δ system. The direct product of two finite groups embedded in a Δ system. The classes which constitute a Δ system may be regarded as a new set Σ' of marks. Since the product of two classes in a Δ system is a unique class in the Δ system, the product of two marks in Σ' is a unique mark in Σ', and the system of product classes is merely the set Ω' of all classes each of which consists of exactly one mark of Σ'. Hence postulates I and II hold for Ω'. If the Δ system does not contain Ω, then by postulate III′ for the Δ system it is true that postulate III holds for Ω'. Hence the Γ system of the Δ system Σ' is precisely Σ', that is, a Marty hypergroup in which multiplication is unique. This is a group, as Marty proved, if and only if in postulate III′ for the Δ system it is true that every S' is unique or every S'' is unique. Hence the first part of Theorem 7 is proved.

THEOREM 7. *If a Δ system does not contain its Marty hypergroup Ω, then the Γ system of this Δ system is this Δ system, and hence a Marty hypergroup in which multiplication is unique. If a Δ system contains Ω but is not equal to Ω, then the Γ system of this Δ system is this Δ system if and only if postulate III_0 holds for this Δ system. If a Δ system is Ω, then it is a Marty hypergroup in which multiplication is unique.*

The second statement in Theorem 7 is true, since postulate III holds for Σ' if and only if postulate III_0 holds for the Δ system.

The direct product of two finite groups, G with elements $a_1, \cdots, a_m$ and H with elements $b_1, \cdots, b_n$, is simply isomorphic with a set of classes embedded in a Δ system. Thus let Σ be $a_1, \cdots, a_m, b_1, \cdots, b_n$; it is no limitation to assume that a_1 is the identity of G and b_1 the identity of H, and that as marks a_1 and b_1 are distinct, although G and H might be subgroups in another group as elements of which a_1 and b_1 are the same. Consider the set of all classes each of which contains at least one mark from among $a_1, \cdots, a_m$ and at least one mark from among $b_1, \cdots, b_n$, with no repetition of marks allowed. This set of classes satisfies postulate I with addition defined as stated for the case that no repetition of marks is allowed. The number of these classes is finite. If R and S are two of these classes then RS is defined as the class whose marks are the distinct ones among all products a_ia_j and b_rb_s as a_i ranges over all the marks among $a_1, \cdots, a_m$ appearing in R, a_j ranges similarly over those in S, b_r ranges over all the marks among $b_1, \cdots, b_n$ appearing in R, and b_s ranges similarly over those in S. Postulate II is satisfied. In fact multiplication is commutative. This system of classes does not contain Ω. It is seen as follows that postulate III′ is satisfied. Let R and S be arbitrary classes. Let a and b be two arbitrary, fixed marks in S. For each mark a_i in S there is a unique mark a_j such that $aa_j = a_i$; for each mark b_r in S there is a unique mark b_s such that $bb_s = b_r$. Define S' as the set whose marks are these a_j and these b_s as a_i and b_r range over R. Then S' is such that $SS' \supset R$. It does not follow that S' is unique. Similarly a class S'' can be found.

Hence this set of distinct classes is a finite Δ system which does not contain Ω. The subset of all classes $\{a, b\}$ is simply isomorphic to the direct product of G and H. By Theorem 7 the Γ system of this Δ system is this Δ system; this Γ system is a Marty hypergroup containing the direct product group.

8. Definition of a Σ algebra; a hypergroup algebra. Consider a set of classes satisfying postulates I and II, but not necessarily postulate III or III′,

that is, a set of classes which is a product system but not necessarily a Δ system. Let the product system contain Σ, by the process described at the end of section 2. Hence every class in the system is a sum of marks in Σ. Hence if, for every class R in the system (and hence in particular for every mark in Σ) and for every positive integer k, the notation kR means $R + \cdots + R$ in which R is a summand k times, and if $Rk = kR$, then every class in the system is a sum, in one and only one way, with coefficients which are positive integers, of marks in Σ. In particular, the product of two marks in Σ is such a sum. Hence the marks in Σ can be used as the basis of an algebra over an arbitrary field, for which the multiplication table of the basis elements is the multiplication table of the marks in Σ in the product system. This algebra may be called a Σ algebra. If the product system satisfies postulate III, then by section 4 the Σ algebra is an algebra whose basis elements form a hypergroup. Conversely, if an algebra is defined with the elements of a hypergroup as basis elements, then this algebra is such a Σ algebra. Such a Σ algebra may be called a hypergroup algebra. If the product system is in fact merely a group then the Σ algebra becomes a group algebra, since Σ is in fact a group.

The properties of elements of algebras are therefore in particular properties of marks in Σ, and hence in particular of elements of a hypergroup. For example, the matrix representation of hypergroups, as presented in section 5 of the paper by Wall, is a very special instance of a standard elementary property of algebras.[5]

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[5] M. Deuring, *Algebren*, Berlin 1935, p. 2; or B. L. v. d. Waerden, *Moderne Algebra*, Berlin 1931, vol. II, p. 131.

MATRICES NORMAL WITH RESPECT TO AN HERMITIAN MATRIX.*

By John Williamson.

Introduction. A square matrix A with elements in the complex number field is said to be a normal matrix, if it is commutative with its conjugate transposed matrix, i. e., if

$$AA^* = A^*A,$$

where $A^* = \bar{A}'$ is the conjugate transposed of A. In particular every hermitian matrix and every unitary matrix is a normal matrix. A necessary and sufficient condition, that a matrix A be a normal matrix, is that A be equivalent under a unitary transformation to a diagonal matrix. In other words a matrix A is normal, if, and only if, there exists a unitary matrix V, such that

$$VAV^* = D, \tag{1}$$

where D is a diagonal matrix, i. e. a matrix all of whose elements not in the leading diagonal are zero.[1]

If $d_1, d_2, \cdots, d_m$ are the distinct latent roots or characteristic numbers of D and ϕ_i is the principal idempotent element of D associated with d_i, $i = 1, 2, \cdots, m$,[2]

$$D = d_1\phi_1 + d_2\phi_2 + \cdots d_m\phi_m.$$

Since ϕ_i is a diagonal matrix, all of whose elements are zero or unity, $\phi^*_i = \phi_i$ and consequently

$$D^* = \bar{d}_1\phi_1 + \bar{d}_2\phi_2 + \cdots \bar{d}_m\phi_m. \tag{2}$$

As ϕ_i is a polynomial in the matrix D, so that $\phi_i = \phi_i(D)$, it is a consequence of (2) that

$$D^* = \bar{d}_1\phi_1(D) + \bar{d}_2\phi_2(D) + \cdots + \bar{d}_m\phi_m(D) = g(D), \tag{3}$$

and of (1) and (3), that

$$A^* = g(A), \tag{4}$$

* Received February 1, 1938.

[1] Aurel Wintner, *Spektraltheorie der unendlichen Matrizen* (1929), p. 24.

[2] J. M. Wedderburn, "Lectures on matrices," *Colloquium Publications* (1934), p. 29.

where $g(x)$ is a polynomial in x. Hence, if A is a normal matrix, the conjugate transposed of A is a polynomial in A. Conversely, let $A^* = g(A)$. Then we may so determine the unitary matrix V in (1) that D is a triangle matrix, in which all of the elements below the leading diagonal are zero.[3] Since V is a unitary matrix, $D^* = g(D)$ and consequently each element of D^*, which lies below the leading diagonal, is zero. As D^* contains no elements, which are different from zero, above the leading diagonal, D is a diagonal matrix. Hence A is a normal matrix. We have therefore proved,

LEMMA 1. *A necessary and sufficient condition that a matrix A be a normal matrix is that $A^* = g(A)$, where $g(x)$ is a polynomial in x.*

By taking the conjugate transposed of the matrices in (4) we see that

$$A = \bar{g}(A^*) = f(A^*), \tag{5}$$

where, if $g(x) = \sum_{i=0}^{n} a_i x^i$, $f(x) = \bar{g}(x) = \sum_{i=0}^{n} \bar{a}_i x^i$. We have therefore, on combining (4) and (5),

$$A = f\{\bar{f}(A)\} = \bar{g}\{g(A)\}. \tag{6}$$

In particular, if A is hermitian, $f(x) \equiv x$ and, if A is unitary, $f(x) \equiv \phi(x)$, where $\phi(A) = A^{-1}$.

Let A be a normal matrix, so that A satisfies (5) and (6). If P is any non-singular matrix and if

$$B = PAP^{-1},$$

then

$$A = P^{-1}BP \quad \text{and} \quad A^* = P^*B^*(P^{-1})^*.$$

Therefore, as a consequence of (5),

$$P^{-1}BP = f(P^*B^*(P^{-1})^*) = P^*f(B^*)(P^{-1})^*,$$

and accordingly

$$BH = Hf(B^*), \tag{7}$$

where

$$H = PP^*. \tag{8}$$

The matrix H, defined by (8), is a positive definite hermitian matrix. It is now natural to make the following definition: *if H is any non-singular hermitian matrix, and, if $BH = Hf(B^*)$, the matrix B is normal with respect to H.*

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[3] Wintner, *op. cit.*, p. 22.

A matrix normal with respect to the unit matrix is therefore a normal matrix. The matrix $f(B^*)$ is commutative with B^*. Further, if C is commutative with every matrix, which is commutative with B^*, C is a polynomial in B.[4] Hence we have the alternative definition; *if $BH = HC$, where C is commutative with every matrix, which is commutative with B^*, the matrix B is normal with respect to the non-singular hermitian matrix H.*

Let B_1 be any matrix similar to B so that

$$B = R^{-1}B_1R.$$

Then,

$$R^{-1}B_1RH = HR^*f(B^*_1)(R^*)^{-1}$$

and therefore

$$B_1H_1 = H_1f(B^*_1),$$

where

(9) $$H_1 = RHR^*.$$

Consequently we have

Lemma 2. *If B is normal with respect to H and $B = R^{-1}B_1R$, then B_1 is normal with respect to $H_1 = RHR^*$. Moreover, if $BH = Hf(B^*)$, $B_1H_1 = H_1f(B^*_1)$.*

Accordingly the theory of matrices, normal with respect to an hermitian matrix H, is similar to that of matrices, normal with respect to any matrix H_1, which satisfies (9) and which is therefore equivalent to H under a non-singular conjunctive transformation. In particular the theory of matrices normal with respect to a positive definite hermitian matrix is similar to that of normal matrices.

The canonical form of a normal matrix under unitary transformations is exceedingly simple; in fact, as remarked earlier, it is a diagonal matrix. Since a unitary matrix is a conjunctive automorph of the unit matrix, representative of all positive definite hermitian matrices, it is interesting to consider the corresponding problem for matrices normal with respect to an hermitian matrix H, which is not necessarily positive definite. Therefore, in what follows we discuss the problem: what are the possible canonical forms for a matrix B, normal with respect to an hermitian matrix H, under similarity transformations by matrices, which are conjunctive automorphs of H; i. e. what are the possible canonical forms for a matrix

(10) $$C = RBR^{-1}$$

[4] Turnbull and Aitken, *Canonical Matrices*, Blackie and Sons (1932), p. 150.

where

(11) $$RHR^* = H$$

and

$$BH = Hf(B^*).$$

For brevity we shall call two matrices B and C, which satisfy (10) and (11), *H-equivalent*. It is therefore an immediate consequence of Lemma 2 that, if B is normal with respect to H and if B and C are H-equivalent, C is also normal with respect to H.

From the remarks following Lemma 2 it is apparent that it is not to be hoped to obtain satisfactory results, when H is a general hermitian matrix, but only when H is a suitably chosen representative of a class of conjunctively equivalent hermitian matrices. This is in fact what happens and we therefore only determine canonical forms of matrices normal with respect to H under H-equivalent transformations when H is suitably chosen. From the canonical forms we immediately deduce necessary and sufficient conditions that two matrices normal with respect to H be H-equivalent. The particular cases, when $f(A^*) = A^*$ or $f(A^*) = (A^*)^{-1}$, which correspond respectively, in the theory of normal matrices, to A being hermitian or unitary, are considered in section 5. These two cases have already been treated separately as quite unrelated problems. However, the methods employed in their solution are very similar and it was this similarity that suggested the discussion of the more general problem now under consideration.

1. Let A_1 and A_2 be two similar matrices, which are both normal with respect to the same non-singular hermitian matrix H. Then A_1 and A_2 are both similar to the same matrix Q and there therefore exist two non-singular matrices P_1 and P_2 such that

(12) $$A_i = P_i^{-1}QP_i, \qquad (i = 1, 2),$$

and

(13) $$S_i = P_iHP^*_i, \qquad (i = 1, 2).$$

Further by Lemma 2 the matrix Q is normal with respect to both of the hermitian matrices S_1 and S_2. We now prove a theorem which greatly simplifies our problem.

THEOREM 1. *Necessary and sufficient conditions that two matrices A_1 and A_2, both normal with respect to H, be H-equivalent is that there exist a non-singular matrix K such that*

$$KQ = QK \text{ and } KS_1K^* = S_2, \tag{14}$$

where Q, S_1 and S_2 are any matrices, which satisfy (12) *and* (13).

First, let A_1 and A_2 be H-equivalent. Then there exists a non-singular matrix R satisfying

$$RA_1R^{-1} = A_2 \tag{15}$$

and

$$RHR^* = H. \tag{16}$$

Let $K = P_2RP_1^{-1}$. Then

$$\begin{aligned} KQ &= P_2RP_1^{-1}Q = P_2RA_1P_1^{-1} \text{ by } (12), \\ &= P_2A_2RP_1^{-1} \text{ by } (15), \\ &= QP_2RP_1^{-1} \text{ by } (12), \\ &= QK. \end{aligned}$$

Further,

$$\begin{aligned} KS_1K^* &= P_2RP_1^{-1}S_1(P^*_1)^{-1}R^*P^*_2 = P_2RHR^*P^*_2 \text{ by } (13), \\ &= P_2HP^*_2 = S_2 \text{ by } (16) \text{ and } (13). \end{aligned}$$

Conversely, if a matrix K exists, which satisfies (14), and if $R = P_2^{-1}KP_1$, R satisfies (15) and (16) and consequently A_1 and A_2 are H-equivalent.

Hence in determining the possible canonical forms of a matrix A, normal with respect to H, under H-equivalent transformations we may proceed as follows. First we let Q be a suitably chosen canonical form of A under similarity transformations and let S be the corresponding hermitian matrix, conjunctively equivalent to H, with respect to which Q is normal. Then we determine a canonical form for S under conjunctive transformations by matrices which are commutative with Q. In other words we need only determine a canonical matrix W such that

$$KSK^* = W \text{ and } KQ = QK. \tag{17}$$

We shall call such a transformation (17) by the matrix K an *admissible* transformation.

2. As mentioned at the end of the previous section we may take Q to be any matrix similar to A. We therefore choose Q to be the *diagonal block* matrix

$$Q = [Q_1, Q_2, \cdots, Q_k], \tag{18}$$

where the latent roots of Q_i are all equal to λ_i, and $\lambda_i \neq \lambda_j$, if $i \neq j$. Since A is normal with respect to H, by Lemma 2, Q is normal with respect to S, so that

$$QS = Sf(Q^*) = SM, \tag{19}$$

where M is the diagonal block matrix

$$M = [M_1, M_2, \cdots, M_k]$$

and

$$M_i = f(Q^*_i), \qquad (i = 1, 2, \cdots, k).$$

Let

$$S = (S_{ij}), \qquad (i, j = 1, 2, \cdots, k),$$

be a *partition* of S *similar* to that of Q in (18); i. e. S_{ij} is a matrix with the same number of rows as Q_i and the same number of columns as Q_j. It now follows from (19) that

$$Q_i S_{ij} = S_{ij} M_j, \qquad (i, j = 1, 2, \cdots, k). \tag{20}$$

The latent roots of M_j, being $f(\bar{\lambda}_j)$, are all the same. Hence, either no latent root of M_j is the same as a latent root of Q_i or all latent roots of M_j are the same as those of Q_i. Since, when $i \neq j$, $\lambda_i \neq \lambda_j$, for a fixed value of i in (20) there is at most one value of j, for which the latent roots of M_j are the same as those of Q_i. But there must be at least one value of j, for which the latent roots of M_j coincide with those of Q_i, as otherwise S_{ij} would be zero for all values of j and S would be singular. There are therefore only two possibilities:

I. The latent roots of M_i are the same as the latent roots of Q_i, $S_{ij} = 0$ when $j \neq i$, and

$$Q_i S_{ii} = S_{ii} M_i. \tag{21}$$

Since S is non-singular the matrix S_{ii} in (21) is non-singular.

II. The latent roots of M_j are the same as those of Q_i but i is different from j. Then $S_{ih} = 0$, if $h \neq j$. Since S is hermitian

$$S_{hi} = S^*_{ih} = 0, \qquad (h \neq j);$$

and since S is non-singular S_{ij} and S_{ji} are both non-singular. In place of (21) there are the two equations

$$Q_i S_{ij} = S_{ij} M_j \quad \text{and} \quad Q_j S_{ji} = S_{ji} M_i.$$

Since S_{ij} and S_{ji} are non-singular the latent roots of M_j are all equal to λ_i and those of M_i are all equal to λ_j.

Accordingly after a re-arrangement of the rows and the same re-arrangement of the columns of the matrices S, M and Q, we see that S becomes a diagonal block matrix. The blocks are of two distinct types;

I. $\qquad S_{ii} = S^*_{ii}, \quad Q_i S_{ii} = S_{ii} M_i, \quad M_i = f(Q^*_i)$

and

$$\text{II.}\qquad \begin{pmatrix} 0 & S_{ij} \\ S_{ji} & 0 \end{pmatrix}, \quad \begin{pmatrix} Q_i & 0 \\ 0 & Q_j \end{pmatrix}\begin{pmatrix} 0 & S_{ij} \\ S_{ji} & 0 \end{pmatrix} = \begin{pmatrix} 0 & S_{ij} \\ S_{ji} & 0 \end{pmatrix}\begin{pmatrix} M_i & 0 \\ 0 & M_j \end{pmatrix},$$

$$S_{ij} = S^*{}_{ji}.$$

It so happens that blocks of type II are very much simpler than those of type I. Accordingly we first consider the reduction of type II. Let

$$X = \begin{pmatrix} E & 0 \\ 0 & S_{ji}^{-1} \end{pmatrix},$$

where E is the unit matrix of the same order as S_{ij}. Then, since

$$X[Q_i, Q_j]X^{-1} = [Q_i, S_{ji}^{-1}Q_jS_{ji}] = [Q_i, M_i] = [Q_i, f(Q^*{}_i)],$$

$[Q_i, Q_j]$ is similar to $[Q_i, f(Q^*{}_i)]$. Further

$$X\begin{pmatrix} 0 & S_{ij} \\ S_{ji} & 0 \end{pmatrix}X^* = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}.$$

Since, in Theorem 1, Q is any matrix similar to A, in Q we may replace $[Q_i, Q_j]$ by $[Q_i, f(Q^*{}_i)]$. If this is done, S is a diagonal block matrix and the block $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ takes the place of the block $\begin{pmatrix} 0 & S_{ij} \\ S_{ji} & 0 \end{pmatrix}$. Hence we have

RESULT a. *If A is normal with respect to H, so that $AH = Hf(A^*)$, and, if there are two latent roots λ_i and λ_j of A, such that $\lambda_j = f(\bar{\lambda}_i)$, then $\lambda_i = f(\bar{\lambda}_j)$. The diagonal block matrix Q contains the block $[Q_i, f(Q^*{}_i)]$ and the corresponding block of S is* $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$.

If λ_i is any latent root of A and $\lambda_i \neq f(\bar{\lambda}_i)$, there must exist a latent root λ_j of A such that

$$\lambda_j = f(\bar{\lambda}_i), \quad \lambda_i = f(\bar{\lambda}_j). \tag{22}$$

Hence, if, for no latent root λ_h of A, is $\lambda_h = f(\bar{\lambda}_h)$, corresponding to each latent root λ_i of A there is a latent root λ_j, such that (22) is true. Then, as all blocks of Q are of type II, it follows that A is of order $2m$ and from result a that H has signature zero. By re-arranging the order of the blocks Q_i in (18) we obtain a matrix F, which is similar to Q and has the form

$$F = [F_1, f(F^*{}_1)],$$

where F_1 is a square matrix of order m. If Q is replaced by F, and this, by Theorem 1, is always possible,

$$S = \begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix},$$

where E_m is the unit matrix of order m. We have therefore proved

THEOREM 2. *Let A be normal with respect to H and let $AH = Hf(A^*)$. If $\lambda_i \neq f(\bar{\lambda}_i)$ for all latent roots λ_i of A, then there exists a non-singular matrix R such that $RAR^{-1} = [F_1, f(F^*_1)]$ and $RHR^* = \begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix}$. The matrices A and H are accordingly both of even order and the signature of H is zero.*

The matrix F_1 is not unique but may be replaced by any matrix similar to it. If, however, F_1 is taken in the classical canonical form, the matrix $[F_1, f(F^*_1)]$ is uniquely determined by A. Accordingly, if A is normal with respect to $S = \begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix}$ and, if no latent root λ_i of A satisfies the equation $f(\bar{\lambda}_i) = \lambda_i$, A is S-equivalent to a unique canonical matrix $[F, f(F^*_1)]$. As a consequence we have

COROLLARY 1. *If A satisfies the hypotheses of Theorem 2 and A is similar to the matrix B, which is also normal with respect to H, then A and B are H-equivalent.*

3. Before considering a further reduction of the blocks of type I we prove three lemmas, the first of which is

LEMMA 3. *Let $Q = [Q_1, Q_2]$, $M = [M_1, M_2]$, $S = (S_{ij})$, $(i, j = 1, 2)$, be similar partitions of three matrices Q, M and S, which satisfy (19). If S is hermitian and S_1 is non-singular, there exists a non-singular matrix K such that $KQ = QK$ and $KSK^* = [S_{11}, \sigma_{22}]$.*

As a consequence of (20)

$$S_{21}S_{11}^{-1}Q_1 = S_{21}M_1S_{11}^{-1} = Q_2S_{21}S_{11}^{-1}.$$

Hence, if $K = \begin{pmatrix} E_1 & 0 \\ -S_{21}S_{11}^{-1} & E_2 \end{pmatrix}$, where E_i is the unit matrix of the same order as Q_i, $KQ = QK$. Further,

$$KSK^* = \begin{pmatrix} E_1 & 0 \\ -S_{21}S_{11}^{-1} & E_2 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} E_1 & S_{11}^{-1}S_{12} \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} - S_{21}S_{11}^{-1}S_{12} \end{pmatrix},$$

and the lemma is proved.

Let U be the auxiliary unit matrix of order n and V the auxiliary unit matrix of order m.[5] We now consider the matrix equation

$$\phi(U)D = D\psi(V), \tag{23}$$

where

$$\phi(x) = \sum_{a=1}^{n-1} b_a x^a, \qquad \psi(x) = \sum_{\beta=1}^{m-1} c_\beta x^\beta \text{ and } b_1 c_1 \neq 0.$$

If $D = (d_{ij})$ $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$, (23) implies

$$\sum_{a=1}^{n-1} b_a d_{a+i,j} = \sum_{\beta=1}^{m-1} c_\beta d_{i,j-\beta}, \qquad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m). \tag{24}$$

It is of course to be understood that in (24) $d_{a+i,j} = 0$, if $\alpha + i > n$, and that $d_{i,j-\beta} = 0$, if $j - \beta \leq 0$.

Let us now suppose that $d_{rs} = 0$ for all values of r and s, for which $s - r \leqq k$. Then, if $j - i = k + 2$, equation (24) becomes

$$b_1 d_{1+i,j} = c_1 d_{i,j-1}, \tag{25}$$

or, when $j = 1$,

$$b_1 d_{1+i,1} = c_1 d_{i0} = 0,$$

so that $d_{1+i,1} = 0$. However (25) is valid only when $i > 0$ and, therefore, if $j = 1$, only when $j - i \leq 0$, i. e. when $k \leq -2$. Hence, if $k \leq -2$, $d_{rs} = 0$ for all values of r and s, for which $s - r \leq k + 1$. On the other hand, if $i = n$, (25) becomes

$$c_1 d_{n,j-1} = b_1 d_{n+1,j} = 0,$$

so that $d_{n,j-1} = 0$. Since $j - n = k + 2$ and $j \leq m$, $k \leq m - n + 2$. Hence, if $k \leq m - n - 2$, $d_{rs} = 0$ for all values of r and s, for which $s - r \leq k + 1$. Therefore, by induction, $d_{rs} = 0$, if $s - r \leq$ {maximum of $m - n - 1$ and -1}. When $m > n$, $0 \leq m - n - 1$, so that $d_{11} = 0$ and the first column of D is zero; when $m < n$, $m - n \leq -1$ and the last row of D is zero. We have therefore

LEMMA 4. *If D is a matrix satisfying (23), the first column of D is zero, when $m > n$, and the last row is zero, when $m < n$. If $m = n$, D is non-singular if, and only if d_{nn} is different from zero.*

If D is a square matrix, so that $m = n$, it follows from the above considerations that,

$$D = D_0 + D_1 + D_2 + \cdots + D_{n-1}, \tag{26}$$

[5] Turnbull and Aitken, *op. cit.*, p. 142.

where the only non-zero elements of D_j are those d_{rs} for which $s-r=j$. Equation (23) may now be written in the form

$$\sum_{\alpha=1}^{n-1} b_\alpha U^\alpha \sum_{j=0}^{n-1} D_j = \sum_{j=0}^{n-1} D_j \sum_{\beta=1}^{n-1} c_\beta U^\beta. \tag{27}$$

Equation (27) and the nature of the matrices D_j imply

$$\sum_{\alpha=1}^{s} b_\alpha U^\alpha D_{s-\alpha} = \sum_{\alpha=1}^{s} c_\alpha D_{s-\alpha} U^\alpha, \qquad (s=1,2,\cdots,n-1). \tag{28}$$

If $D_0 = \epsilon E$ and $D_1 = D_2 = \cdots = D_{r-1} = 0$, it is a consequence of the first r of the equations (28) that

$$b_i = c_i, \qquad (i=1,2,\cdots,r). \tag{29}$$

If in addition $b_{r+1+j} = c_{r+j} = 0$ for all $j > 0$, the $(r+1)$-th equation in (28) becomes

$$b_1 U D_r + \epsilon b_{r+1} U^{r+1} = b_1 D_r U. \tag{30}$$

When (30) is satisfied, the remaining of the equations (28) may be solved successively for $D_{r+1}, \cdots, D_{n-1}$.[6] We have therefore

LEMMA 5. *If equation* (30) *is satisfied there exists a matrix*

$$D = \epsilon E + D_r + D_{r+1} + \cdots + D_{n-1}$$

such that

$$D \sum_{\alpha=1}^{r-1} b_\alpha U^\alpha D^{-1} = \sum_{\alpha=1}^{r+1} b_\alpha U^\alpha.$$

4. The matrices in a block of type I satisfy (21) where the latent roots of Q_i all have the same value λ_i. Accordingly we temporarily drop all suffixes i or, what is equivalent to this, assume that all the latent roots of A have the same value λ. We may therefore take Q in the classical canonical form

$$Q = [Q_1, Q_2, \cdots, Q_k],$$

where

$$Q_i = \lambda(E_i + U_i), \lambda \neq 0; \qquad Q_i = U_i, \lambda = 0. \tag{31}$$

In (31) E_i is the unit matrix of order e_i and U_i the auxiliary unit matrix of the same order. The elementary divisors of $A - xE$ are therefore $(x-\lambda)^{e_i}$, $(i=1,2,\cdots,k)$. We further suppose that $e_1 \geqq e_2 \geqq \cdots \geqq e_k$. The matrix $M = [M_1, M_2, \cdots, M_k]$, where

[6] H. W. Turnbull, "Power vectors," *Proceedings of the London Mathematical Society*, Series 2, vol. 39 (1934), part 2, pp. 106-146.

(32) $$M_i = \lambda(E_i + \sum_{j=1}^{e_i-1} a_j U'_i{}^j),\ \lambda \neq 0; \qquad M_i = \sum_{j=1}^{e_i-1} a_j U'_i{}^j,\ \lambda = 0.$$

Since $M_i = f(Q^*_i)$, the a_j in (32) are determined uniquely by the polynomial $f(x)$ and are independent of i.

Let

$$S = (S_{ij}), \qquad (i, j = 1, 2, \cdots, k),$$

be a partition of S similar to that of Q. Then as a consequence of (20) we have

(33) $$U_i S_{ij} = S_{ij} \sum_{k=1}^{e_j-1} a_k U'_j{}^k, \qquad (i, j = 1, 2, \cdots, k).$$

Equations (33) are the same whether λ is or is not zero. If T_j is the secondary unit matrix of order e_j [7]

$$T_j{}^2 = E_j \quad \text{and} \quad T_j U'_j = U_j T_j.$$

Hence, if

$$T = [T_1, T_2, \cdots, T_k]$$

and

$$S = DT = (D_{ij}T_j), \qquad (i, j = 1, 2, \cdots, k),$$

equations (33) become

$$U_i D_{ij} T_j = D_{ij} T_j \sum_{k=1}^{e_j-1} a_k U'_j{}^k = D_{ij} \sum_{k=1}^{e_j-1} a_k U_j{}^k T_j$$

or

(34) $$U_i D_{ij} = D_{ij} \sum_{k=1}^{e_j-1} a_k U_j{}^k.$$

Since M is similar to Q, $a_1 \neq 0$ and equations (34) are all of the type (23). If $e_1 > e_2$, by Lemma 4 the last row of D_{1j} is zero except when $j = 1$. Since S is non-singular D is non-singular and the last row of D_{11} is not zero. The only element in the last row of D_{11}, which is different from zero, is the element in the last column and therefore, by Lemma 4, D_{11} is non-singular. If $e_1 = e_2 = \cdots = e_c > e_{c+1}$ and D_{ii} is non-singular for some value of i, $1 \leq i \leq c$, by a suitable re-arrangement of the rows of D and the same re-arrangement of the columns we may move D_{ii} into the place of D_{11} without disturbing Q or M. Accordingly we may suppose in this case that D_{11} is non-singular. There only remains the case, in which D_{ii} is singular for all values of i, $1 \leq i \leq c$. Let d_{ij} be the element in the last row and the last column of D_{ij}. Then

(35) $$d_{ii} = 0, \qquad (i = 1, 2, \cdots, c).$$

[7] Turnbull and Aitken, *op. cit.*, p. 11.

Since D is non-singular, for at least one value of i, $1 < i \leq c$, $d_{1i} \neq 0$, as otherwise one row of D would be zero. Hence without any loss of generality we may assume that $d_{12} \neq 0$. Let E be the unit matrix of order e_1 and ϵ be a complex number. Then

$$\begin{pmatrix} E & \epsilon E \\ 0 & E \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} E & \epsilon E \\ 0 & E \end{pmatrix}^* = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix},$$

where $K_{11} = D_{11} + \epsilon D_{21} + \bar{\epsilon} D_{12} + \epsilon\bar{\epsilon} D_{22}$. The element in the last row and last column of K_{11} is $k_{11} = d_{11} + \epsilon d_{21} + \bar{\epsilon} d_{12} + \epsilon\bar{\epsilon} d_{22} = \epsilon d_{21} + \bar{\epsilon} d_{12}$ by (35). Since ϵ is arbitrary, for at least one value of ϵ, $k_{11} \neq 0$. The matrix $\begin{pmatrix} E & \epsilon E \\ 0 & E \end{pmatrix}$ is commutative with $[Q_1, Q_2]$ and therefore the above transformation is admissible. Hence by Lemma 4, if $k_{11} \neq 0$, K_{11} is non-singular. We may suppose then that such a transformation, if necessary, has been made, and that D_{11} and therefore S_{11} is non-singular. By Lemma 3 there is therefore an admissible transformation which reduces S to the form $[S_{11}, \sigma]$. By repetitions of the above process we finally reduce S by an admissible transformation to the diagonal block form

$$S = [S_1, S_2, \cdots, S_k],$$

where S_i is of the same order as Q_i and

$$Q_i S_i = S_i f(Q^*{}_i) = S_i M_i. \tag{36}$$

If $A - xE$ has the single elementary divisor $(x - \lambda)^n$, (19) coincides with (36). Hence we need now only consider this particular case, in which

$$Q = \lambda(E + U),\ \lambda \neq 0; \qquad Q = U,\ \lambda = 0,$$

$$M = \lambda(E + \sum_{j=1}^{n-1} a_j U'^j),\ \lambda \neq 0; \qquad M = \sum_{j=1}^{n-1} a_j U'^j,\ \lambda = 0,$$

and

$$S = DT$$

where D is given by (26). The equations that correspond to (24) are

$$d_{i+1,j} = \sum_{\beta=1}^{n-1} a_\beta d_{i,j-\beta}$$

and in particular, when $j = i + 1$,

$$d_{i+1,i+1} = a_1 d_{ii}. \tag{37}$$

Since S is hermitian, $D_j T$ is hermitian and in particular so is $D_0 T$. Therefore

$$d_{n-i,n-i} = \bar{d}_{i+1,i+1}. \tag{38}$$

As a consequence of (37) and (38) a_1 is of unit modulus. If $n = 2m + 1$ and $i = m$, (38) becomes

$$d_{m+1,m+1} = \bar{d}_{m+1,m+1} = \epsilon\rho^2$$

where ρ is real and $\epsilon = \pm 1$. It now follows from (37) that

$$D_0 = \epsilon\rho^2[a_1^{-m}, a_1^{-(m-1)}, \cdots, a_1^{-1}, 1, a_1, \cdots, a_1^{m-1}, a_1^{m}].$$

Let b be a particular one of the square roots of a_1 and K be the matrix

$$K = \rho^{-1}[b^m, b^{m-1}, \cdots, b, 1, b^{-1}, \cdots, b^{-m}].$$

Then, since $b = \sqrt{a_1}$, and $|a_1| = 1$, $b^{-1} = \bar{b}$ and consequently $KT = TK^*$. Therefore,

$$KD_0TK^* = KD_0KT = \epsilon T, \qquad \epsilon = \pm 1, \tag{39}$$

while a simple calculation shows that

$$KUK^{-1} = bU. \tag{40}$$

If, however, $n = 2m$, (37) and (38) imply that $\bar{d}_{mm} = d_{m+1,m+1}$. Hence, if $d_{m+1,m+1} = \rho^2 e^{i\theta}$, where ρ is real, $a_1 = e^{-2i\theta}$. Since $e^{i\theta}$ is determined by d_{mm}, if b is a definite square root of a_1, $e^{i\theta} = \epsilon b$, $\epsilon = \pm 1$. Therefore

$$D_0 = \epsilon\rho^2[b^{-(2m-1)}, b^{-(2m-3)}, \cdots, b^{-1}, b, \cdots b^{(2m-3)}, b^{(2m-1)}].$$

If

$$K = \rho^{-1}[b^{(2m-1)/2}, \cdots, b^{\frac{1}{2}}, b^{-\frac{1}{2}}, \cdots, b^{-[(2m-1)/2]}],$$

equations (39) and (40) are again valid. Moreover as a consequence of (40),

$$(K^{-1})^*U'K^* = \bar{b}U' = U'/b. \tag{41}$$

We now prove

LEMMA 6. *There exists a non-singular matrix R such that*

$$RUR^{-1} = \sum_{i=1}^{n-1} b_i U^i \quad \text{and} \quad RSTR^* = \epsilon T,$$

where $b_1 = b$ and $\epsilon = \pm 1$.

We shall prove this lemma by induction and therefore assume that there exists a matrix W such that

$$WUW^{-1} = \sum_{i=1}^{r} b_i U^i \quad \text{and} \quad WSW = DT, \tag{42}$$

where D is given by (26) and

(43) $$D_0 = \epsilon D, \qquad D_1 = D_2 = \cdots = D_{r-1} = 0.$$

We first note that the matrix K, which satisfies (39) and (40), is a matrix W satisfying (42), when $r = 1$. Since $US = SM$, where $M = \sum_{j=1}^{n-1} a_j U'^j$,

$$WUW^{-1}WSW^* = WSW^*(W^*)^{-1}MW$$

and by (42)

$$\sum_{i=1}^{r} b_i U^i DT = DT \sum_{i=1}^{n-1} c_i U'^i$$

or finally

(44) $$\sum_{i=1}^{r} b_i U^i D = D \sum_{i=1}^{n-1} c_i U^i.$$

Equation (44) is of the form (23) and it follows from (43) that (29) is true and

(45) $$b_1 U D_r = b_1 D_r U + \epsilon c_{r+1} U^{r+1}.$$

If $b_{r+1} = c_{r+1}/2$, (44) becomes

$$b_1 U \frac{-D_r}{2} + \epsilon b_{r+1} U^{r+1} = b_1 \frac{-D_r}{2} U.$$

This last equation is the same as (30), if D_r is replaced by $-D_r/2$. Therefore by Lemma 5 we can determine matrices $F_{r+1}, F_{r+2}, \cdots, F_{n-1}$, where F_j has the same form as D_j, such that the matrix

$$N = \epsilon E - D_r/2 + F_{r+1} + \cdots + F_{n-1}$$

satisfies

(46) $$N \sum_{i=1}^{r} b_i U^i N^{-1} = \sum_{i=1}^{r+1} b_i U^i.$$

Since DT is hermitian $D_r T = T D^*_r$, and therefore

(47) $$NDTN^* = (\epsilon E - D_r/2 + F_{r+1} + \cdots + F_{n-1}) \times D(\epsilon E - D_r/2 + G_{r+1} + \cdots + G_{n-1})T,$$

where G_j has the same form as D_j. On multiplying out the right hand side of (47) we find that

(48) $$NDTN^* = D_0 + H_{r+1} + \cdots + H_{n-1},$$

where H_j is of the same form as D_j. Equations (42), (46) and (48) imply

that (42) and (43) are true when r is replaced by $r+1$ and W by NW. Our induction is now complete and the lemma proved.

By (45)

$$(UD_r - D_rU)T = \frac{\epsilon c_{r+1}}{b_1} U^{r+1}T.$$

Since D_rT is hermitian,

$$\begin{aligned}\{(UD_r - D_rU)T\}^* &= TD^*_rU' - TU'D^*_r = D_rUT - UD_rT \\ &= -(UD_r - D_rU)T.\end{aligned}$$

Hence $\frac{\epsilon c_{r+1}}{b}T$ is anti-hermitian and therefore c_{r+1}/b is a pure imaginary quantity. Since $b_{r+1} = c_{r+1}/2$ we have

$$b_{r+1} = ib_1\beta_{r+1}, \tag{49}$$

where β_{r+1} is real. Further, since the c_i in (44) are determined uniquely, by the polynomial $f(x)$, in terms of $b_1, b_2, \cdots, b_r$ it follows that the b_j in Lemma 6 are determined uniquely, apart from the sign of b_1, by the polynomial $f(x)$. If R is the matrix of Lemma 6,

$$RQR^{-1} = \lambda(E + \sum_{i=1}^{n-1} b_iU^i),\ \lambda \neq 0, \qquad RQR^{-1} = \sum_{i=1}^{n-1} b_iU^i,\ \lambda = 0,$$

$$(R^*)^{-1}MR^* = \lambda(E + \sum_{i=1}^{n-1} b_iU^{i\prime}),\ \lambda \neq 0, \qquad (R^*)^{-1}MR^* = \sum_{i=1}^{n-1} b_iU^{i\prime},\ \lambda = 0,$$

$$RSR^* = \epsilon T.$$

We have now proved

RESULT b. *Let A be normal with respect to H, so that $AH = Hf(A^*)$ and let λ_j be a latent root of A such that $f(\bar{\lambda}_j) = \lambda_j$. If $(x - \lambda_j)^{e_k}$ is an elementary divisor of $A - xE$, the matrix A is similar to a diagonal block matrix Q, which contains the block*

$$\begin{aligned} Q_{jk} &= \lambda_j(E_k + b_{1j}U_k + ib_{1j}\sum_{r=2}^{e_k-1} \beta_{rj}U_k^r),\ \lambda_j \neq 0, \\ &= b_{1j}U_k + ib_{1j}\sum_{j=2}^{e_k-1} \beta_{rj}U_k^r,\ \lambda_j = 0. \end{aligned} \tag{50}$$

The corresponding block in the matrix S, with respect to which Q is normal, is $\epsilon_{jk}T_k$, where $\epsilon_{jk} = \pm 1$. In (50) b_{1j} is of modulus one and each β_{rj} is real.

By combining results a and b we have

THEOREM 3. *If A is normal with respect to H, there exists a non-singular matrix P, such that $PAP^{-1} = Q$ and $PHP^* = S$, where Q and S are diagonal*

block matrices. Corresponding to each pair of latent roots of type II Q contains a block $[Q_i, f(Q^*_i)]$ *given in result a and S the block* $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$; *corresponding to each elementary divisor* $(x - \lambda_j)^{e_k}$ *of type I Q contains the block* Q_{jk} *given by* (50) *and S the block* $\epsilon_{jk} T_k$.

The matrices Q and S of Theorem 3 are canonical forms for A and H. Apart from the ϵ_{jk}, which have the value ± 1, everything in the canonical forms is uniquely determined by A, H and $f(x)$. If $(x - \lambda_j)^{e_k}$ occurs exactly t times among the elementary divisors of $A - xE$, with this elementary divisor are associated t ϵ_{jk} each of which has the value ± 1. The number of positive ϵ_{jk} is called the *index* associated with the elementary divisor $(x - \lambda_j)^{e_k}$. We now justify this terminology by showing that the index associated with each elementary divisor, as defined above, is uniquely determined by the matrices A and H.

Let Q, S_1 and Q, S_2 be two canonical forms for A and H. Then by Theorem 1, there exists a non-singular matrix K commutative with Q such that $KS_1K^* = S_2$. Since K is commutative with Q, K is a diagonal block matrix partitioned similarly to Q in (18). Consequently we need only consider the case in which all latent roots of Q are the same and of type II, since S_1 coincides with S_2 as far as blocks of type I are concerned. Let

$$Q = [Q_1, Q_2, \cdots, Q_k], \quad S_1 = [\epsilon_1 T_1, \epsilon_2 T_2, \cdots, \epsilon_k T_k], \quad S_2 = [\rho_1 T_1, \rho_2 T_2, \cdots, \rho_k T_k],$$

where $\epsilon_i = \pm 1$, $\rho_i = \pm 1$ and Q_j is of the form (50). If

$$K = (K_{ij}), \qquad (i, j = 1, 2, \cdots, k),$$

is a partition of K similar to that of Q,

$$\sum_{a=1}^{k} K_{ia} \epsilon_a T_a K^*_{ja} = \delta_{ij} \rho_i T_i, \tag{51}$$

where δ_{ij} is the Kronecker δ. If k_{ij} is the element in the top left-hand corner of K_{ij} and $e_{c-1} > e_c = e_{c+1} = \cdots = e_d > e_{d+1}$, (51) implies [8]

$$\sum_{a=c}^{d} k_{ia} \epsilon_a \bar{k}_{ja} = \delta_{ij} \rho_i, \qquad (i, j = c, c+1, \cdots, d). \tag{52}$$

Since K is non-singular, $|k_{ij}| \neq 0$, $(i, j = c, c+1, \cdots, d)$ and by (52) $[\epsilon_c, \epsilon_{c+1}, \cdots, \epsilon_d]$ is conjunctively equivalent to $[\rho_c, \rho_{c+1}, \cdots, \rho_d]$. Hence the

[8] John Williamson, "The equivalence of non-singular pencils of hermitian matrices in an arbitrary field," *American Journal of Mathematics*, vol. 57 (1935), pp. 484-485.

index associated with $(x-\lambda)^{e c}$ in the canonical form Q, S_1 is the same as that in the form Q, S_2. Consequently we have

THEOREM 4. *If A and B are normal with respect to H, A and B are H-equivalent if, and only if, the two matrices $A-xE$ and $B-xE$ have the same elementary divisors and if the indices associated with each elementary divisor of type I are the same for both matrices.*

If H is positive definite each elementary divisor is of type I and is linear while each index has the value one. We have therefore the well known

COROLLARY 1. *Two normal matrices which are similar are equivalent under a unitary transformation.*

5. Special cases. Let $f(x)=x$, so that $f(A^*)=A^*$. A latent root of type I must now be real. If λ_j is real, we have from (50)

$$Q^*{}_{jk}=\lambda_j(E_k+\bar{b}_{1j}U'_k-i\bar{b}_{1j}\sum_{r=2}^{e_k-1}\beta_{rj}U'_k{}^r)=\lambda_j(E_k+b_{1j}U'_k+ib_{1j}\sum_{r=2}^{e_k-1}\beta_{rj}U'_k{}^r).$$

Therefore b_{1j} is real and $\beta_{rj}=0$. Since b_{1j} has modulus one $b_{1j}=\pm 1$ and can therefore be taken as $+1$. The matrix

$$Q_{jk}=\lambda_j E_k+\lambda_j U_k.$$

Since $AH=HA^*=(AH)^*$, the matrix $H_1=AH$ is hermitian. If $PAP^{-1}=Q$ and $PHP^*=S$, $PH_1P^*=S_1=QS$. The matrices S and $S_1=QS$ are therefore canonical forms for the pair of hermitian matrices H and H_1 under conjunctive transformations. From Theorem 4 we can therefore deduce necessary and sufficient conditions for the conjunctive equivalence of two pairs of hermitian matrices.[9] If $f(x)=-x$, a latent root λ_j of type I must be pure imaginary and $b_{1j}=i$, $\beta_{rj}=0$. This could of course have been deduced from the previous case, since if $A=A^*$, $(iA)^*=-iA$.

Let $f(A^*)=(A^*)^{-1}$, so that A is a conjunctive automorph of H. If a latent root λ_j is of type I, $\lambda_j=-1/\bar{\lambda}_j$, so that $\lambda_j\bar{\lambda}_j=1$ or $\lambda_j=e^{i\theta}$. On dropping all suffixes j we have from (50)

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[9] H. W. Turnbull, "On the equivalence of pencils of hermitian forms," *Proceedings of the London Mathematical Society*, vol. 39 (1935), pp. 232-248; M. H. Ingraham and K. W. Wegner, "The equivalence of pairs of hermitian matrices," *Transactions of the American Mathematical Society*, vol. 38 (1935), pp. 145-162; G. R. Trott, "On the canonical form of a non-singular pencil of hermitian matrices," *American Journal of Mathematics*, vol. 56 (1936), no. 3, pp. 359-391. The methods used in Trott's paper are very similar to those of this paper.

$$Q_{jk} = e^{i\theta}(E + b_1 U + b_1 i \sum_{r=2}^{e-1} \beta_r U^r) \tag{53}$$

and

$$f(Q^*_{jk}) = (Q^*_{jk})^{-1} = e^{i\theta}(E + b_1 U' + b_1 i \sum_{r=2}^{e-1} \beta_r U'^r).$$

Therefore

$$(Q_{jk})^{-1} = e^{-i\theta}(E + \bar{b}_1 U - \bar{b}_1 i \sum_{r=2}^{e-1} \beta_r U^r). \tag{54}$$

On multiplying (53) and (54), the coefficient of U on the right is $b_1 + \bar{b}_1$ and on the left is zero. Hence $b_1 = i$ and we have

$$E = U^2 + (E - \sum_{r=2}^{e-1} \beta_r U^r)^2. \tag{55}$$

If $z = \sum_{r=2}^{e-1} \beta_r U^r$ and $U = x$, (55) may be written in the form

$$(1 - z)^2 = 1 - x^2. \tag{56}$$

Therefore

$$(1 - z) = (1 - x^2)^{\frac{1}{2}},$$

$$z = \tfrac{1}{2}U^2 + \tfrac{1}{8}U^4 + \cdots 1 \cdot 3 \cdot 5 \cdots (2k - 3) U^{2k}/2^k k! + \cdots. \tag{57}$$

Consequently (53) becomes

$$Q_{jk} = e^{i\theta}(E + iU - \tfrac{1}{2}U^2 - \tfrac{1}{8}U^4 - \cdots - 1 \cdot 3 \cdot 5 \cdots (2k-3) U^{2k}/2^k k! + \cdots. \tag{58}$$

The canonical form obtained from Theorem 3 by giving Q_{jk} the value (58) serves as an alternative to one found in a previous paper for the particular case now under consideration.[10]

6. If A is normal with respect to H so that $AH = Hf(A^*)$, the polynomial $f(x)$ is, as remarked in the introduction, by no means a general polynomial. If A has no latent root of type I, the nature of $f(x)$ can be deduced from Theorem 2. We now consider the nature of $f(x)$ when $A - xE$ has the single elementary divisor $(\lambda - x)^n$. By (50)

$$Q = \lambda(E + b_1 U + b_1 i \sum_{r=2}^{n-1} \beta_r U^r)$$

and

$$f(Q^*) = \lambda(E + b_1 U' + b_1 i \sum_{r=2}^{n-1} \beta_r U'^r)$$

and

[10] John Williamson, "Quasi-unitary matrices," *Duke Mathematical Journal*, vol. 3 (1937), no. 4, pp. 720-722.

$$Q^* = \bar{\lambda}(E + \bar{b}_1 U' - \bar{b}_1 i \sum_{r=2}^{n-1} \beta_r U'^r).$$

If $\dfrac{Q^* - \bar{\lambda}E}{\bar{\lambda}\bar{b}_1} = \eta$,

$$\frac{f(Q^*) - \lambda E}{\lambda b_1} = 2U' - \eta \tag{59}$$

while

$$\eta = U' - i \sum_{r=2}^{n-1} \beta_r U'^r. \tag{60}$$

Equation (60) may be solved for U' as a power series in η by an inductive process and therefore by (59), $f(Q^*)$ is determined as a polynomial in Q^*. If all the latent roots of A have the same value λ, $f(Q^*)$ remains unchanged even if $A - xE$ has more than one elementary divisor. The general case follows from the above by replacing $\bar{\lambda}\bar{b}_1\eta$ by the principal nilpotent element of Q^* associated with λ.[11]

7. The matrix S in the canonical form of Theorem 3 is determined by A and H and not by H alone. The blocks of S are of two types

$$\text{Type I} \quad \epsilon T_j; \qquad\qquad \text{Type II} \quad \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}.$$

It is possible to determine conjunctive transformations which reduce both types to diagonal form and therefore to find a matrix W such that

$$WSW^* = [E_s, -E_t],$$

where E_j is the unit matrix of order j and $s - t$ is the signature of H or S. The matrix

$$WQW^{-1} = R$$

can then be calculated. The matrices R thus obtained give unique canonical forms for all matrices normal with respect to $[E_s, -E_t]$ under similarity transformations by matrices which are conjunctive automorphs of $[E_s, -E_t]$. If t or s is zero, we obtain the result from which we started; every normal matrix can be reduced to diagonal form by a unitary transformation. We have not carried out this reduction as the canonical form R is not nearly as simple as that of Q in Theorem 3.[12]

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[11] Wedderburn, *op. cit.*, p. 29.

[12] Reductions of the type mentioned have been carried out in special cases. See (10) and John Williamson, "On the normal forms of linear canonical transformations in dynamics," *American Journal of Mathematics*, vol. 59 (1937), no. 3, pp. 614-617.

SUBRINGS OF DIRECT SUMS.*

By Neal H. McCoy.

1. Introduction. If K is any ring, a direct sum of rings K is understood to be the ring of all functions with values in K, defined on some finite or infinite set M. In a recent paper [1] the following criterion was established for determining when a ring is isomorphic to a subring of a direct sum. *A necessary and sufficient condition that a ring R be isomorphic to a subring of a direct sum of rings K is that for every element $a \neq 0$ in R there exists a homomorphism h of R into a subring of K such that $h(a) \neq 0$.*

In the paper referred to, this theorem was used to show that if every element a of a commutative ring R satisfies the conditions $a^p = a$ and $pa = 0$, where p is a fixed prime, then R is isomorphic to a subring of a direct sum of Galois fields $GF(p)$. This is, in itself, a generalization of a theorem concerning Boolean rings, proved by Stone in a different manner.[2] The primary purpose of the present paper is to establish several extensions of these results as well as some other theorems of a related nature. The method used in establishing the existence of the necessary homomorphisms is believed to be somewhat simpler than those used previously.

2. Preliminary definitions. Let R be a given ring, and a any non-zero element of R. If there exists a positive integer n such that $na = 0$, the least such n will be called the *order* of a, this coinciding with the usual definition of order in the additive group of the ring. If every element of R is of order n, then n must clearly be a prime p, and we may say that R has the *characteristic*

* Received June 25, 1937.

[1] N. H. McCoy and Deane Montgomery, "A representation of generalized Boolean rings," to appear in the *Duke Mathematical Journal*. I am indebted to Professor Montgomery for several helpful suggestions during the preparation of the present paper.

Note added February 23, 1938. The paper just referred to has appeared in the *Duke Mathematical Journal*, vol. 3 (1937), pp. 455-459. In a more recent paper, which has been submitted to the same journal, it is shown that every commutative ring without nilpotent elements is isomorphic to a subring of a direct sum of fields. However, this general result does not give much or any information as to just what fields are involved in such a direct sum representation of a given ring. It will be noted that Theorems 2, 3 and 5 of the present paper furnish this information for the case of suitably restricted rings.

[2] M. H. Stone, "The theory of representations for Boolean algebras," *Transactions of the American Mathematical Society*, vol. 40 (1936), pp. 37-111.

p. If no element of R has finite order, we say that R has *characteristic* 0. In the sequel we shall refer to a ring of characteristic k, it being understood that k is either 0 or a prime p.

The commutative ring R will be called an *algebraic ring* if every element $a \neq 0$ of R satisfies an equation of the type

$$f_a(x) \equiv m_0x^n + m_1x^{n-1} + \cdots = 0 \qquad (m_0a^n \neq 0), \tag{1}$$

the coefficients m_i being integers. If R has no unit element, it is of course assumed that this equation has no constant term.

3. Imbedding theorem.

We now prove the following theorem.

THEOREM 1. *An algebraic ring R of characteristic k, without a unit element, may be imbedded in an algebraic ring R' of characteristic k with a unit element. If R contains no nilpotent elements, then R' contains no nilpotent elements.*

The proof is perhaps best divided into two cases according as $k = 0$ or $k = p$. Suppose first that $k = 0$. Consider the ring consisting of the pairs (a, n), where a is in R and n is an integer, with addition and multiplication defined as follows:[3]

$$(a, n) + (b, m) = (a + b, n + m),$$
$$(a, n)(b, m) = (ab + nb + ma, nm).$$

By $(a, n) = (b, m)$ we mean that $a = b$ and $n = m$. It follows at once that this is a ring of characteristic 0, with unit element $(0, 1)$, and with a subring isomorphic to R, namely the subring consisting of those elements of the form $(a, 0)$. It will be noted that this procedure shows the possibility of adjoining to R a ring element e having the following properties: (i) $ae = ea = a$ for all elements a of R, (ii) $e^2 = e$, and (iii) $a + ne = b + me$ implies $a = b$, $n = m$. The desired ring R' is then the ring $R[e]$ whose elements are of the form $a + ne$. Clearly e is the unit element of R'.

If $f(x) = 0$ is the equation (1) satisfied by a, then by Taylor's theorem we find that

$$f(a) = f(-ne) + f'(-ne)(a + ne)$$
$$+ (1/2!)f''(-ne)(a + ne)^2 + \cdots = 0.$$

Thus an arbitrary element $a + ne$ of R' satisfies an algebraic equation with

[3] J. L. Dorroh, "Concerning adjunctions to algebras," *Bulletin of the American Mathematical Society*, vol. 38 (1932), pp. 85-88.

integral coefficients, and R' is an algebraic ring. If R has no nilpotent elements, clearly no element $a + ne$ of R' can be nilpotent, and the theorem is established for this case.

Suppose now that R is of characteristic p, and let the elements of $GF(p)$ be denoted by $\bar{0}, \bar{1}, \cdots, \overline{p-1}$. Consider the ring of pairs $(a, \bar{n})$, where a is in R and $\bar{n}$ is in $GF(p)$, with addition and multiplication defined as follows:

$$(a, \bar{n}) + (b, \bar{m}) = (a + b, \bar{n} + \bar{m}),$$
$$(a, \bar{n})(b, \bar{m}) = (ab + nb + ma, \bar{n}\bar{m}).$$

This ring is of characteristic p, with unit element $(0, \bar{1})$, and containing the subring of elements $(a, \bar{0})$ which is isomorphic to R. We may now proceed in a manner entirely analogous to that used above. The details will therefore be omitted.

4. Principal theorems on subrings of direct sums. Let P_k denote the prime field of characteristic k. Thus P_0 will be the field of rational numbers, and P_p will denote the $GF(p)$. Let C_k be the essentially unique algebraically closed, algebraic field in which P_k can be imbedded.[4] We shall now prove the following theorem.

THEOREM 2. *Every algebraic ring R of characteristic k without nilpotent elements is isomorphic to a subring of a direct sum of fields C_k.*

In view of the preceding theorem, it is sufficient to limit ourselves to the case in which R has a unit element e. Let I denote the subring of R consisting of the integral multiples of e. Thus if $k = 0$, I is isomorphic to the ring of rational integers, while if $k = p$, I is isomorphic to $P_p = GF(p)$. We may clearly consider the coefficients in $f_a(x)$ given by (1) as elements of I.

If D is any ring, we denote by $D[x]$ the ring of polynomials in an indeterminate x with coefficients in D. We now establish the following lemma.

LEMMA 1. *Let D be a subring of C_k, and M an ideal in $D[x]$ containing no non-zero element of D. Then there exists an element ρ of C_k such that $g(\rho) = 0$ provided $g(x) \equiv 0(M)$.*

Since D contains no divisors of zero, it may be imbedded in a quotient field $D' \subset C_k$.[5] Let $h(x)$ be an element of M of minimum degree $(\geqq 1)$. If $g(x)$ is in M, then in $D'[x]$ we may write

$$g(x) = q(x)h(x) + r(x),$$

[4] See B. L. van der Waerden, *Moderne Algebra* I, p. 199.

[5] van der Waerden, *op. cit.*, p. 46.

the degree of $r(x)$ being less than the degree of $h(x)$. This equation may now be multiplied by a properly chosen non-zero element d of D such that $dq(x)$ and $dr(x)$ have coefficients in D. It then follows that $dr(x) \equiv 0(M)$, and being of degree less than the degree of $h(x)$ therefore vanishes identically. Thus

$$dg(x) = dq(x)h(x).$$

and for the ρ of the lemma we only need to choose an arbitrary root in C_k of the equation $h(x) = 0$.

Before proceeding to the proof of the theorem, we establish another lemma. If S is a subring of R, and a an element of R not in S, then by $S[a]$ we mean the ring generated by elements of S together with a, that is the ring of polynomials in a with coefficients in S.

LEMMA 2. *Let S be a subring of R containing the unit element e of R, and h a given homomorphism $S \to D$, where D is a subring of C_k. If a is an element of R not in S, then the homomorphism h may be extended*[6] *to a homomorphism $h' : S[a] \to D_1$, where $D \subset D_1 \subset C_k$.*

The elements of $S[a]$ are expressible as polynomials in a with coefficients in S. Suppose by the given homomorphism h, an arbitrary element s of S corresponds to the element $\bar{s}$ of D. To the element $\Sigma s_i a^i$ of $S[a]$ we may now make correspond the element $\Sigma \bar{s}_i x^i$ of $D[x]$. The set of all polynomials of $D[x]$ which thus correspond to the zero element of $S[a]$ forms an ideal M in $D[x]$. A polynomial $\Sigma \bar{s}_i x^i$ thus belongs to M if and only if there exist elements s_i of S such that $\Sigma s_i a^i = 0$ and $s_i \to \bar{s}_i$ by h. We proceed to show that M contains no non-zero element of D.

We wish to show that if $\sum_{k=0}^{m} s_k a^k = 0$, $\bar{s}_k = 0$ $(k \neq 0)$, then $\bar{s}_0 = 0$. For convenience, let us denote $\sum_{k=0}^{m} s_k a^k$ by A. From the equation (1) satisfied by a, it follows that

$$m_0 a^n = -(m_n e + \cdots + m_1 a^{n-1}).$$

Thus $m_0{}^2 a^{n+1}$, $m_0{}^3 a^{n+2}, \cdots$ are also expressible as linear combinations of $e, a, \cdots, a^{n-1}$, with integral coefficients. Hence if l is a positive integer $> m - n$, we may write

$$m_0{}^{l+i} a^i A = \sum_{k=0}^{m} s_k m_0{}^{l+i} a^{k+i} \qquad (i = 0, 1, \cdots, n-1),$$

[6] By this we mean that under h' the elements of S have the same image as under the given homomorphism h.

then replace $m_0^{l+i}a^{k+i}$ (for $k+i \geqq n$) by linear combinations of $e, a, \cdots, a^{n-1}$, and collect coefficients of the different powers of a. We are thus led to the following equations:

$$(2) \qquad m_0^{l+i}a^iA = \sum_{j=0}^{n-1} b_{ij}a^j = 0 \qquad (i = 0, 1, \cdots, n-1),$$

where the b_{ij} are linear combinations of the s_k with integral coefficients. From the method of formation of the b_{ij}, it follows that $b_{ii} = m_0^{l+i}s_0 + b'_{ii}$, where b'_{ii} is a linear combination of the s_k ($k \neq 0$). Also, if $i \neq j$, b_{ij} is a linear combination of the s_k ($k \neq 0$). It follows that

$$(3) \qquad | b_{ij} | = m_0^u s_0^n + B,$$

where u is a positive integer and B is a homogeneous polynomial in the s_k with integral coefficients, every term of which contains at least one s_k other than s_0.

Now let c_i be the co-factor of b_{i0} in $| b_{ij} |$, multiply the i-th equation (2) by c_i and add, thus showing that $| b_{ij} | = 0$. By the homomorphism h, it follows that $m_0^u \bar{s}_0^n + \bar{B} = 0$. But since $\bar{s}_k = 0$ ($k \neq 0$), it is evident from the form of B that $\bar{B} = 0$. Thus $m_0^u \bar{s}_0^n = 0$, and hence $\bar{s}_0 = 0$.

We are now in a position to complete the proof of the lemma. By the first lemma, there exists an element ρ of C_k such that $g(\rho) = 0$ provided $g(x) \equiv 0(M)$. The correspondence

$$(4) \qquad \Sigma\, s_i a^i \rightarrow \Sigma\, \bar{s}_i \rho^i$$

then defines the required homomorphism $h' : S[a] \rightarrow D[\rho] = D_1$. For if $\Sigma\, s_i a^i = \Sigma\, r_i a^i$, then $\Sigma(\bar{s}_i - \bar{r}_i)x^i \equiv 0(M)$, $\Sigma\, \bar{s}_i \rho^i = \Sigma\, \bar{r}_i \rho^i$, and thus the correspondence (4) is actually independent of the manner in which an element of $S[a]$ is expressed as a polynomial in a with coefficients in S. It then follows readily that (4) defines a homomorphism, and in fact an extension of h as required.

In view of the theorem stated in the introduction, we shall establish Theorem 2 by showing the existence of a homomorphism of R into a subring of C_k taking any prescribed element $a \neq 0$ of R into a non-zero element of C_k.

Let $\{e, a\}$ denote the subring of R generated by a and e, the elements are therefore polynomials in a with coefficients in I. The set of polynomials $\Sigma\, n_i x^i$ in $I[x]$ such that $\Sigma\, n_i a^i = 0$ is an ideal N in $I[x]$ which clearly contains no non-zero element of I. We remark also that N contains no element of the form nx^m ($ne \neq 0$). For, if the contrary were true, then $na^m = 0$ and since it is assumed that R has no nilpotent element it would follow that

$a = 0$. Thus there exists, by Lemma 1, an element $\sigma \neq 0$ of C_k such that if $g(x) \equiv 0(N)$, then $g(\sigma) = 0$. It now follows that the correspondence

$$\Sigma n_i a^i \rightarrow \Sigma n_i \sigma^i$$

is a homomorphism h of $\{e, a\}$ into $I[\sigma] \subset C_k$, by which $a \rightarrow \sigma \neq 0$. Except for showing that we may choose $\sigma \neq 0$, it will be noted that this is merely a special case of Lemma 2 in which S is the ring I.

It is now clear how to proceed. We assume that the elements of R are well ordered, with the first two elements as e and a respectively, where a is an arbitrary element of R other than zero. Let $\{e, a\}$ be the S of Lemma 2 and suppose b is the first element of R not in $\{e, a\}$. Then the homomorphism $h\colon \{e, a\} \rightarrow I[\sigma]$ can be extended to a homomorphism $\{e, a, b\} \rightarrow I[\sigma, \delta] \subset C_k$. We now may apply the lemma again with $\{e, a, b\}$ replacing the S. By transfinite induction it is readily shown that h can be extended to a homomorphism of R into a subring of C_k.[7] There thus exists a homomorphism of R into a subring of C_k which takes the arbitrary element $a \neq 0$ of R into an element $\sigma \neq 0$ of C_k. The proof of the theorem is therefore completed.

It will be noted that the ρ introduced in the proof of Lemma 2 is a root of the equation $f_a(x) = 0$, since $f_a(a) = 0$ and therefore $f_a(x) \equiv 0(M)$. Let us now assume that R is of characteristic p, and that *every* element of R satisfies the equation $x^{p^n} - x = 0$. It is readily verified that every element of the ring R' introduced in Theorem 1 also satisfies this same equation. By the above construction of a homomorphism of R into a subring of C_p, the only subrings of C_p which enter are those obtainable from $I = GF(p)$ by adjunction of roots of the equation $x^{p^n} - x = 0$. But the roots in C_p of this equation are precisely the elements of the Galois field $GF(p^n)$. We have thus established the following theorem.

THEOREM 3. *If the commutative ring R has characteristic p and every element satisfies the equation $x^{p^n} - x = 0$, then R is isomorphic to a subring of a direct sum of $GF(p^n)$.*

In the statement of this theorem it is not necessary to assume that R has no nilpotent elements, as it is readily verified that $a^m = 0$ is incompatible with $a^{p^n} = a$ unless $a = 0$.

This theorem will be seen to be a generalization of a result obtained in the joint paper with Montgomery, referred to in the introduction. In par-

[7] See Stone, *loc. cit.*, p. 102.

ticular, if we choose $p=2$, $n=1$, this is equivalent to the theorem of Stone which states that a Boolean ring is isomorphic to an algebra of subclasses of some class, addition of classes being carried out modulo 2.

5. Rings with all elements of finite order. Let S be an arbitrary ring all of whose elements are of finite order, and denote by S_p the set of elements of S whose orders are powers of the prime p. It follows easily that S_p is an ideal in S and that if $p \neq q$, S_p and S_q have no element except zero in common. From this it follows that the product of an element of S_p by one of S_q is always zero.

Let a be an arbitrary element of S, and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ its order. Then the numbers $n/p_1^{\alpha_1}, \cdots, n/p_k^{\alpha_k}$ are relatively prime and there therefore exist integers $\beta_1, \cdots, \beta_k$ such that

$$\beta_1 n/p_1^{\alpha_1} + \cdots + \beta_k n/p_k^{\alpha_k} = 1.$$

From this it follows that if $a_{p_i} = (\beta_i n/p_i^{\alpha_i})a$, then a_{p_i} is an element of S_{p_i} and $a = \sum_{i=1}^{k} a_{p_i}$. Thus each element of S is expressible as a sum of a finite number of elements belonging respectively to S_p $(p = 2, 3, 5, \cdots)$. In fact it may be verified that each element can be so expressed in a unique way.

Now let P be the set of all primes p. In accordance with the concept of direct sum used above we shall define *the direct sum* of the rings S_p $(p = 2, 3, 5, \cdots)$ to be the ring of all functions defined on P, such that on p the values of the functions are in S_p. In this connection we have the following theorem.

THEOREM 4. *A ring S all of whose elements are of finite order is isomorphic to a subring of the direct sum of the ideals S_p $(p = 2, 3, 5, \cdots)$.*[8]

To establish this theorem we need only to exhibit an isomorphic ring of functions of the required type. If a is an arbitrary element of S we make correspond to it the function $f_a(p)$ defined as follows. If p is not a divisor of the order of a, then $f_a(p) = 0$, while if p is a divisor of the order of a, then $f_a(p) = a_p$ as defined above. The set of all functions f_a, where a ranges

[8] It will be noted that S is in fact isomorphic to the ring of all functions defined on P, such that on p the functional values are in S_p, with the restriction that all functions differ from zero on only a finite number of points of P. From our point of view S is therefore a proper subring of the direct sum of the S_p although from other points of view S might well be called the direct sum of the S_p. See, e. g., Leo Zippin, "Countable torsion groups," *Annals of Mathematics*, vol. 36 (1935), pp. 86-99.

over the elements of S, is readily shown to be the required ring. For the discussion above shows that $f_{ab}(p) = f_a(p)f_b(p)$ and $f_{a+b}(p) = f_a(p) + f_b(p)$. That the correspondence $a \rightarrow f_a$ is actually an isomorphism follows from the fact that if $a \neq 0$ then $f_a(p) \neq 0$ for some p.

Let us now assume that S has no nilpotent elements. Then the order of no element can have a square factor. For if $k^2la = 0$, $kla \neq 0$, then $(kla)^2 = 0$, contrary to the assumption that there are no nilpotent elements. It follows that the ring S_p has characteristic p. From Theorems 2 and 4 we then obtain the following result.

THEOREM 5. *An algebraic ring R without nilpotent elements, all of whose elements are of finite order, is isomorphic to a subring of a direct sum of fields C_p, where p ranges over the primes.*

6. Ideal arithmetic in rings of characteristic p. We conclude with a brief study of ideal arithmetic in a ring of characteristic p. Let R be an algebraic ring of characteristic p with no nilpotent elements. If a is an arbitrary element of R, the subring $\{a\}$ generated by a is a finite commutative ring without nilpotent elements, and is therefore a direct sum of finite fields of characteristic p, say $GF(p^{n_i})$ $(i = 1, 2, \cdots, k)$.[9] If n is the L. C. M. of the n_i, then every element of $\{a\}$, in particular a itself, satisfies the equation $x^{p^n} - x = 0$.

Let N be an ideal in R, other than the unit ideal R itself, and consider the quotient ring R/N. This ring is a commutative ring of characteristic p, every element of which satisfies an equation $x^{p^n} - x = 0$, for proper choice of n. It follows that R/N contains no nilpotent elements and thus by Theorem 2, there is a homomorphism of R/N into a subring of C_p which takes any prescribed non-zero element of R/N into a non-zero element of C_p. We may now prove the following theorem.[10]

THEOREM 6. *Let R be an algebraic ring of characteristic p with no nilpotent elements. If M and N are ideals in R such that N is not a divisor of M, then there exists a prime ideal P in R which is a divisor of N but not of M.*

Let a be an element of R which is in M but not in N, and denote by $\bar{a}$ the image of a under the homomorphism $R \rightarrow R/N$. Thus $\bar{a} \neq 0$, and by the above discussion there exists a homomorphism $R/N \rightarrow C'_p$, where C'_p is a subring of C_p, taking $\bar{a}$ into an element of C_p different from zero. Thus the

[9] van der Waerden, *op. cit.*, II, p. 163.

[10] Stone, *loc. cit.*, p. 105.

correspondence $R \rightarrow R/N \rightarrow C'_p$ defines a homomorphism of R into C'_p taking a into a non-zero element. This means, however, that there exists an ideal P in R such that $R/P \cong C'_p$. Since $C'_p \subset C_p$ has no divisors of zero, it follows that P is a prime ideal.

It is now easy to establish the final theorem.[10]

THEOREM 7. *Let R be an algebraic ring of characteristic p with no nilpotent elements. Then in R every ideal other than the unit ideal is the product of all its prime ideal divisors.*

Let $K \neq R$ be an ideal in R. Then K is not a divisor of R, and by the preceding theorem there exists a prime ideal containing K, so that K actually has prime ideal divisors. Let L be the product (intersection) of all the prime ideal divisors of K. Then clearly $K \subset L$. If L were not contained in K, there would be a prime ideal containing K but not L, contrary to the definition of L. Thus $L \subset K$, from which we conclude that $L = K$, and the theorem is proven.

SMITH COLLEGE.

METABELIAN GROUPS AND TRILINEAR FORMS.*

By Robert M. Thrall.

1. Introduction. H. R. Brahana has shown [1] that every metabelian group, G, composed of operators all except identity of prime order p, which contains H (an abelian group of order p^m and type $1, 1, 1, \cdots$) as a maximal invariant abelian subgroup and is generated by H and m independent permutable operators $u_1, \cdots, u_m$ from the group of isomorphisms of H, determines a trilinear form with coefficients in the $GF[p]$:

$$F(x, y, z) = \sum_{h,i,j} a_{hij} x_h y_i z_j,$$

and that conversely every such trilinear form determines a metabelian group having the above properties. We may suppose that G is generated by

$$H = \{s_1, \cdots, s_k, r_1, \cdots, r_l, t_1, \cdots, t_{n-k-l}\}$$

and $U = \{u_1, \cdots, u_m\}$ where $\{r_1, \cdots, r_l, t_1, \cdots, t_{n-k-l}\}$ is the central, $K = \{r_1, \cdots, r_l\}$ is the commutator subgroup, and $S = \{s_1, \cdots, s_k\}$ has no operators invariant under U. Then G is the direct product of $G' = \{S, U\}$ and $T = \{t_1, \cdots, t_{n-k-l}\}$. It is evident that classification of the groups G' implies the classification of all the groups G; so it is sufficient to consider only the groups G', and, henceforth we shall do so, (but will for simplicity in notation drop the primes).

The trilinear form $F(x, y, z)$ which has for a_{hij} the exponent of r_h in the commutator of s_j and u_i is completely determined by the choice of generators of K, H, and U. A different choice of generators would in general determine a different form F'. For instance, if new generators $u'_1, \cdots, u'_m$ of U are defined by the relations $u_i = u'^{\alpha_{i1}}_1 \cdot \cdots \cdot u'^{\alpha_{im}}_m$ $(i = 1, \cdots, m)$, F' would be obtained from F by applying the transformation

$$y'_i = \alpha_{1i} y_1 + \cdots + \alpha_{mi} y_m \qquad (i = 1, \cdots, m)$$

i. e. by applying the contragredient transformation on the y_i. Similarly changes of generators in S and in k would imply the contragredient transformations on the z_j and x_h respectively.

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* Presented to the Society, April 9, 1937. Received by the Editors May 24, 1937.

[1] *Duke Mathematical Journal*, vol. 1 (1935), pp. 185-197. The results of this article will be used throughout the introduction without further reference.

Given a three-way matrix $M(a_{hij})$ there are six ways in which sets of variables x, y, z can be associated with the elements a_{hij} to give trilinear forms, these changes corresponding to interchanging the rôles played by the sets of variables x, y, z in the form $F(x, y, z)$. Each of these six interpretations of the matrix defines a group G. Of these six groups not more than three are abstractly distinct, since in G the subgroups S and U play abstractly the same rôles and hence can be interchanged without giving a new abstract group G. However, if l, m, k are all different at least three of the six groups G are distinct, since then there are commutator subgroups of three different orders: p^l, p^m, p^k.

2. Equivalence of trilinear forms and isomorphism of related groups G. We shall say that two forms $F_1(x, y, z)$ and $F_2(x, y, z)$ are *conjugate* or *equivalent* if, and only if, it is possible to transform F_1 into F_2 by means of linear transformations with coefficients in the $GF[p]$ on x, y, and z separately. A form $F(x, y, z)$ together with all the other forms conjugate to it will be said to constitute a *class of equivalent forms.* If F_1 and F_2 belong to the same class we write $F_1 \sim F_2$.

If G_1 is defined by $F_1(x, y, z)$ and G_2 is defined by $F_2(x, y, z)$, then G_1 is simply isomorphic to G_2 if either $F_2(x, y, z) \sim F_1(x, y, z)$ or

$$F_2(x, y, z) \sim F'_1(x, y, z) \equiv F_1(x, z, y)$$

where F'_1 has as its *h-matrices* [2] those of F_1 with the rows and columns interchanged. We define τ as the operation of interchanging y and z or in terms of the groups as the operation of interchanging S and U. If either $F_1 \sim F(x, y, z)$ or $F_1 \sim F'(x, y, z) \equiv F(x, z, y)$ we will write $F_1 \overset{\tau}{\sim} F$ and the totality of forms so related to a given one will be called a *τ-class.* It is evident that the groups corresponding to the forms of a τ-class are all simply isomorphic, and conversely if G_1 is simply isomorphic to G_2 and if the isomorphism can be exhibited by new choices of generators in S, U, and K and perhaps also use of τ, then the forms F_1 and F_2 belong to the same τ-class.

Two groups G_1 and G_2 are simply isomorphic if, and only if, generators $\{s'_1, \cdots, s'_k, u'_1, \cdots, u'_m\}$ of G_2 and $\{r'_1, \cdots, r'_l\}$ of K_2 can be chosen so that if $s_j^{-1}u_i^{-1}s_ju_i = \prod_{h=1}^{l} r_h^{a_h}$ in G_1 then $s'^{-1}_j u'^{-1}_i s'_j u'_i = \Pi r'^{a_h}_h$, i. e., if generators of G_2 and K_2 can be so chosen that the trilinear forms of G_1 and G_2 are the same. We associate with a given group G a collection E of forms, one form for each set of generators of G which satisfy our initial conditions as to

[2] *Ibid.*, p. 190.

maximal subgroups etc. Two groups are simply isomorphic if, and only if, the two collections which they define are the same.[3] From the preceding discussion it is clear that if one member of a τ-class D is in a collection E, then the whole τ-class is in E. If D coincides with E for a group G, then G is only isomorphic to other groups having the same τ-class. However, in general D does not coincide with E. For the numbers m and k previously defined, considered as an unordered pair, are invariants of a τ-class but are not necessarily invariants of the group G which defines the τ-class. For instance, the group $G = \{S, U\}$ of order p^{2+4+4} defined by $s_1^{-1}u_1^{-1}s_1u_1 = r_1$, $s_1^{-1}u_2^{-1}s_1u_2 = r_2$, $s_2^{-1}u_3^{-1}s_2u_3 = r_3$, $s_2^{-1}u_4^{-1}s_2u_4 = r_4$; the operators otherwise permutable, is likewise generated by the subgroups $S' = \{s_1, u_3, u_4\}$ and $U' = \{s_2, u_1, u_2\}$ which also satisfy the hypotheses of the first paragraph (§ 1). From the first definition of G we get $m = 2$, $k = 4$ whereas from the second we get $m = k = 3$. But in any case a collection E can be written as a sum of (distinct) τ-classes, so a determination of τ-classes is a first step toward the classification of groups G. We proceed along this line postponing the more general problem of classification of collections E.

We first consider the changes induced in the matrix $M(a_{hij})$ when the variables x, y, and z undergo separate linear transformations. Let 1) $z_j = \Sigma \alpha_{j\nu} z'_\nu$ $(j = 1, 2, \cdots, k)$. Then

$$F(x, y, z) = \sum_{h,i,j} a_{hij} x_h y_i z_j = \sum_{h,i,\nu} a'_{hi\nu} x_h y_i z'_\nu$$

where

$$a'_{hi\nu} = \sum_j a_{hij} \alpha_{j\nu}.$$

Now consider the two dimensional matrix

$$M_x = (\sum_h a_{hij} x_h).$$

Under 1) this matrix is replaced by $M_x \cdot (\alpha_{j\nu})$. Similarly if 2) $y_i = \sum_\mu \beta_{\mu i} y'_\mu$ $(i = 1, 2, \cdots, m)$, F becomes $F' = \sum_{h,\mu,j} a'_{h\mu j} x_h y'_\mu z_j$ where $a'_{h\mu j} = \sum_i \beta_{\mu i} a_{hij}$ and M_x is replaced by $(\beta_{\mu i}) \cdot M_x$. Evidently M_x completely defines $M(a_{hij})$ and conversely. Under changes in y and z, M_x may be replaced by any matrix PM_xQ, where P and Q, of ranks m and k respectively, are any non-singular square matrices with elements in the $GF[p]$. Under τ the rows and columns of M_x are interchanged. If m and k are not equal, two forms (l, m, k) which

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[3] Brahana's statement (*loc. cit.*, p. 189), concerning isomorphism of groups defined by two forms is incorrect and should be replaced by the above.

belong to different classes will also belong to different τ-classes. Hence, in this case, we need only consider the classes (l, m, k) where, say, $m > k$ to obtain the τ-classes (l, m, k). $M_x = (\sum_h a_{hij}x_h) = \sum_h (a_{hij})x_h$. If we replace x by x' where $x_h = \sum_\lambda \gamma_{h\lambda}x'_\lambda$ we have $M_x = \sum_h (a_{hij}) \cdot \sum_\lambda \gamma_{h\lambda}x'_\lambda = \sum_\lambda (a'_{\lambda ij})x'_\lambda$ where $a'_{\lambda ij} = \sum_h \gamma_{h\lambda}a_{hij}$. That is, the h-sections of $M(a_{hij})$ are replaced by linear combinations of themselves under changes in x. The minors of M_x, polynomials in $x_1, \cdots, x_l$, are replaced by linear combinations of themselves under changes in y and z, and τ. And under changes in x these polynomials are replaced by polynomials conjugate to them under linear transformations on x. Hence, the projective invariants of the "invariant factors" of M_x will be invariants of the class or τ-class to which M_x belongs.

An analogous argument will generalize the results of the two preceding paragraphs to M_y and M_z, the arguments being identical in the case of classes but differing somewhat in the case of τ-classes.

We now generalize the concept of h-sections. We say that M_x for a fixed value a of x (i. e. $x_h = a_h$, $h = 1, 2, \cdots, l$) is an *x-section* of M_x and likewise of $M(a_{hij})$. We denote this by $M_x(a)$. The h-sections are then

$$M_x(1, 0, \cdots, 0), \cdots, M_x(0, 0, \cdots, 1).$$

We note that the rank of an x-section is unchanged under changes on y and z, and τ. But evidently the totality of x-sections is unchanged by linear transformation on x. Hence, the number of x-sections of any given rank is an invariant of M_x and also of $M(a_{hij})$. Similarly the numbers of y-sections or z-sections of any given rank are invariants of $M(a_{hij})$, and of the τ-classes aside from the possibility of interchanging the numbers for y and z under τ.

If $m = k$, $F(x, y, z)$ and $F(x, z, y)$ might be equivalent. Hence, we find the classes with invariants l, m, m, and then combine such of those as are equivalent under τ. Furthermore, for $m = k$ the matrix M_x is square, and its determinant, the only m-rowed minor, is a form $f(x)$ of degree m in the l variables x. The projective invariants of $f(x)$ will be invariants of the τ-class to which M_x belongs. This gives particular interest to the forms and groups having $m = k$, and the major portion of this discussion will be devoted to the case $m = k = 3$.

We propose to classify the groups G and related forms F by classifying the matrices M_x under multiplication on left and right by non-singular matrices with constant elements, linear transformations on x, and τ if $m = k$. This scheme will give each abstract group G once for each τ-class that it defines, and hence each group at least once. In the case $m = k = 3$ we shall

complete the classification by finding the groups that can be defined by more than one τ-class.

That a matrix M_x should correspond to a group G imposes certain restrictions on it. First, there must be one element in each row of M_x different from zero, since no u_i is permutable with every s_j. Similarly there must be one element in each column different from zero. This must be true not only for M_x but also for every M'_x in the same τ-class as M_x. (We say that M_x and M'_x are in the same τ-class if the forms corresponding to them belong to the same τ-class).

Brahana[4] has shown that no one of the numbers l, m, k can be greater than the product of the other two, and that there is just one group when one of the numbers is the product of the other two.

3. Apolarity of trilinear forms. Let us write $r_{ij} = u_i^{-1}s_j^{-1}u_is_j$. The mk commutators r_{ij} generate K, of order p^l, and hence satisfy $mk - l$ independent relations which can be expressed in the form $\prod_{i,j} r_{i,j}^{\alpha_{\nu ij}} = 1$, $\nu = 1, \cdots, mk - l$, or replacing the subscript i, j by $\lambda = (i - 1)k + j$ this becomes $\prod_{\lambda} r_\lambda^{\alpha_{\nu\lambda}} = 1$, $\nu = 1, \cdots, mk - l$. For these relations to be independent the matrix $(\alpha_{\nu\lambda})$ must be of rank $mk - l$. In the group G^* with $l = mk$ the commutators r_{ij} satisfy no relations and can always be taken as a set of independent generators for K. That is, in this group the expression for any operator of K in terms of the r_{ij} is unique.

We shall show that *any group G with K of order p^l is a quotient group of the group G^* with $l = mk$, where p^m and p^k are the orders of U and S in G and G^*.* Consider the quotient group of G^* with respect to an operator $\bar{r} = \prod_{\lambda} r_\lambda^{\beta_\lambda}$. ($\bar{r}$ is in the central since G^* is metabelian). This group $G^*/\bar{r}$ has $l = mk - 1$ and the relation satisfied by its commutators is $\prod_{\lambda} r_\lambda^{\beta_\lambda} = 1$. The quotient group G^*/R where $R = \{\bar{r}_1, \cdots, \bar{r}_{mk-l}\}$, $\bar{r}_\nu = \prod_{\lambda} r_\lambda^{\beta_{\nu\lambda}}$, $\nu = 1, \cdots, mk - l$ has its relations given by the rows of the matrix $(\beta_{\nu\lambda})$. As the $\beta_{\nu\lambda}$ are at our disposal they may be chosen as the $\alpha_{\nu\lambda}$ of the relations connecting the commutators r_λ of the given group G. And as two groups whose commutators $r_\lambda \equiv r_{ij}$ satisfy the same relations are certainly simply isomorphic, the theorem follows. Further, the quotient group G^*/R can be obtained by taking successive quotient groups of index p, i. e. first $G_1 = G^*/\bar{r}_1$ then $G_2 = G_1/\bar{r}_{12}$ and so on until we reach $G_{mk-l} = G^*/R$ (where $\bar{r}_{12}$ in G_1 corresponds to $\bar{r}_2$ in G^* etc.). *Hence any group G with K of order p^l is a quotient*

[4] *Ibid.*, p. 191.

group of at least one group G with K of order p^{l+1} (provided $l < mk$). These successive quotient groups could just as well have been taken with respect to any set of generators of R (and the operators corresponding to them in the successive quotient groups). If these new generators $\bar{r}'_h$ are given in terms of the $\bar{r}_\nu$ by the relations $\bar{r}'_h = \prod_\nu \bar{r}_\nu{}^{\gamma_{h\nu}}$ and hence in terms of the r_λ by the relations

$$\bar{r}'_h = \prod_\nu \left(\prod_\lambda r_\lambda{}^{\alpha_{\nu\lambda}} \right)^{\gamma_{h\nu}} = \prod_\lambda r_\lambda{}^{\alpha'_{h\lambda}}$$

where $(\alpha'_{h\lambda}) = (\gamma_{h\nu})(\alpha_{\nu\lambda})$, the matrix $(\alpha'_{h\lambda})$ defines the same group G as $(\alpha_{\nu\lambda})$. Further, if we let $\alpha_{\nu\lambda} = \alpha_{\nu ij}$ and $\alpha'_{h\lambda} = \alpha'_{hij}$, where as before $\lambda = (i-1)k + j$, then the forms

$$F(x, y, z) = \Sigma\alpha_{\nu ij}x_\nu y_i z_j \quad \text{and} \quad F' = \Sigma\alpha'_{hij}x_h y_i z_j$$

belong to the same τ-class. Choice of a new set of generators in U and S corresponds as before to linear transformations on y and z; so if a form F describes the relations satisfied by the commutators r_{ij} then any other form $F' \overset{\tau}{\sim} F$ will describe the relations of a group G' simply isomorphic with G.

Thus we have the group G defined by two sets of forms: the one giving the relations satisfied by the commutators $r_\lambda \equiv r_{ij}$ and the other, the original one, giving the expression for the r_{ij} in terms of any given set of generators of K. We shall establish certain "apolarity" relations between these two sets of forms (or rather between their matrices) which will cut in half the work necessary for determining the τ-classes, and hence, the groups G, with numbers l, m, k.

Suppose that a given group G is defined in the two above ways by the relations 1) $\prod_{i,j} r_{ij}{}^{b_{\nu ij}} = 1$, $\nu = 1, \cdots, mk - l$, and 2) $r_{ij} = \prod_{h=1}^{l} \bar{r}_h{}^{a_{hij}}$, $i = 1, \cdots, m$; $j = 1, \cdots, k$. Substituting 2) in 1) we have

$$\prod_{i,j} \left(\prod_{h=1}^{l} \bar{r}_h{}^{a_{hij}} \right)^{b_{\nu ij}} = \prod_{h=1}^{l} \bar{r}_h{}^{\sum_{i,j} a_{hij}b_{\nu ij}} = 1.$$

But since the $\bar{r}_h$ are independent this requires 3) $\sum_{i,j} a_{hij}b_{\nu ij} = 0$, $h = 1, \cdots, l$; $\nu = 1, \cdots, mk - l$. Two matrices satisfying 3) will be called *apolar*; likewise the two τ-classes defined by them will be called *apolar*.

Since the apolarity condition is symmetric in the two matrices (a_{hij}) and $(b_{\nu ij})$ we may by interchanging their rôles obtain a new group G' of order $p^{mk-l+m+k}$ which group we will say is *apolar* to the initial group G.[5] It is

[5] If neither a given τ-class nor its apolar τ-class implies that one of $\{S, K\}$ and

evident that the classification of the groups $G(l, m, k)$ implies the classification of the groups $G(mk - l, m, k)$ and hence that for purposes of classification we never need consider l larger than $mk/2$.

4. General theory for the groups and forms $(l, 3, 3)$. We shall summarize the preceding argument by applying it to the case $m = k = 3$, $l = 0, 1, \cdots, 9$. The matrix M_x becomes $M_x = (\sum_{h=1}^{l} a_{hij}x_h)$ where we need consider only $l = 0, 1, 2, 3, 4$. Under changes of generators in U and in S, M_x is replaced by PM_xQ where P and Q are non-singular constant three rowed matrices. Under τ, M_x is replaced by its transposed. Under changes on x the element a_{ij} of M_x is replaced by $a'_{ij} = \sum_\lambda a'_{\lambda ij}x'_\lambda$ where $x_h = \sum_\lambda \alpha_{\lambda h}x'_\lambda$ and $a'_{\lambda ij} = \sum_h \alpha_{\lambda h}a_{hij}$. Further, $|M_x| = f(x)$, a cubic form in $x_1, \cdots, x_l$, is replaced by $f_1(x')$, a form conjugate to $f(x)$ under the given transformation. So, first, we classify cubic forms in l variables. Then choosing a particular form from each projective class we ask how many τ-classes can have this cubic for $|M_x|$. Given two matrices M_x and M'_x with $|M_x| = |M'_x| = f(x)$ the forms F and F' are equivalent if $PM_xQ = M'_x$. If $f(x) \equiv g(x') = cf(x')$ we say that the transformation from x to x' is an *automorphism* of $f(x)$. The totality of such transformations constitutes a group A, called the *group of automorphisms* of $f(x)$. M_x and M'_x are equivalent if, and only if, $PM''_xQ = M'_x$ where M_x is replaced by M''_x under some automorphism of $f(x)$. $M_x \overset{\tau}{\sim} M'_x$ if either $M_x \sim M'_x$ or $M_x \sim M'_x$-transpose. (In what follows we shall be concerned chiefly with M_x and will drop the subscript, using $M(x_1, \cdots, x_l)$ instead of $M_x(x_1, \cdots, x_l)$. We shall continue to use the subscripts in $M_y(y)$ and $M_z(z)$.)

5. $l = 0$. Classification of commutators. For $l = 0$ the only τ-class is the null class. It defines the abelian group $\{S, U\}$ of order p^6 which is not a group G, (see footnote 5), and the group $G^* = \{S, U\}$ of order p^{6+9} which is as we have seen the only group G with the numbers $9, 3, 3$. As the other groups $G(l, 3, 3)$ are all quotient groups of this group G^* with respect to subgroups of K an analysis of the nature of the operators in G^* will throw light upon the group theoretic meaning of the invariants of the τ-classes. The succeeding arguments given for the case $(l, 3, 3)$ are capable of direct generalization to the general case (l, m, k).

$\{U, K\}$ shall not be maximal abelian, the two groups defined by it are both groups G. Otherwise it may happen that one (or both) of these groups will not be in the set of groups G, although it will be the direct product of a group G and an abelian group of order p^a and type $1, 1, 1 \cdots$.

In a group G we say that the commutator of u^y (i. e. $u_1^{y_1}u_2^{y_2}u_3^{y_3}$) and s^z is a *commutator of the first kind.*[6] An operator in K which is not a commutator of the first kind but which is a product of two commutators of the first kind will be called a *commutator of the second kind.* Now any operator, T, in G can be written as $u^y s^z r^x$. The commutator of T and any other operator $T' = u^{y'} s^{z'} r^{x'}$, $T^{-1}T'^{-1}TT'$, is

$$\begin{aligned} r^{-x}s^{-z}u^{-y} \cdot r^{-x'}s^{-z'}u^{-y'} \cdot u^y s^z r^x \cdot u^{y'} s^{z'} r^{x'} &= s^{-z}(u^{-y}s^{-z'}u^y s^{z'})s^{-z'}(u^{-y'}s^z u^{y'} s^{-z})s^z s^{z'} \\ &= (u^{-y}s^{-z'}u^y s^{z'})(u^{-y'}s^z u^{y'} s^{-z}). \end{aligned}$$

Hence, *every commutator in a group G is of kind* 0, 1, *or* 2. In general these do not include all of the operators of K, and we shall say that an operator of K is a *commutator of the ν-th kind* if it is a product of ν but of no less than ν commutators of the first kind. (In a group $G(l, m, k)$ no commutator is of kind greater than the smaller of m and k.)

The general commutator of the first kind in the group $G^*(9, 3, 3)$, that of u^y and s^z, has as its matrix[7]

$$\begin{pmatrix} y_1z_1 & y_1z_2 & y_1z_3 \\ y_2z_1 & y_2z_2 & y_2z_3 \\ y_3z_1 & y_3z_2 & y_3z_3 \end{pmatrix} = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{pmatrix}$$

and hence is of rank one provided that neither u^y nor s^z is identity. Conversely, *any operator in K with a matrix (α_{ij}) of rank one is a commutator of the first kind,* since, then, there exist numbers y_1, y_2, y_3; z_1, z_2, z_3 such that $\alpha_{ij} = y_i \cdot z_j$.

If r^a is a commutator of the second kind in G^*, its matrix (α_{ij}) will be the sum of the matrices for the commutators of the first kind of which it is the product. If (α_{ij}) were of rank less than two, r^a would be of kind less than two, and since the sum of two matrices of ranks k_1 and k_2 is of rank $k_1 + k_2$ at most, we have that *the matrix of a commutator of the second kind in G^* is of rank two.* Conversely, *an operator in K with matrix of rank two is of the second kind.* For if (α_{ij}) is of rank two there exist non-singular matrices P and Q such that $(\alpha_{ij}) = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}Q$, whence

$$(\alpha_{ij}) = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}Q + P\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}Q$$

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[6] Identity is said to be of the zero-th kind.

[7] The element in the i-th row and j-th column is the exponent of r_{ij} in this commutator. Every such matrix defines an element of K.

is an expression of (α_{ij}) as the sum of two matrices of rank one, giving r^a as the product of two commutators of the first kind. Similarly, *a necessary and sufficient condition that an operator in K shall be a commutator of the third kind is that its matrix be of rank three.*[8] In G^* we have therefore as many non-commutators in the commutator subgroup K as there are three rowed matrices of rank three. The ratio of this number to the order of K, approaches the limit one as the prime, p, becomes infinite.[9]

Now consider a commutator $r^a = \prod_{i,j} r_{ij}{}^{a_{ij}}$ in a group $G(mk - l, m, k)$. For $l > 0$ the matrix (a_{ij}) is not uniquely defined by r^a. For identity in G corresponds to the invariant subgroup $R = \{\prod_{i,j} r_{ij}{}^{a_{hij}}\}$, $h = 1, \cdots, l$, in $G^*(mk, m, k)$ and if r^a corresponds to a particular commutator $r'^{a'}$ in G^* then it also corresponds to those and only to those commutators in the coset $Rr'^{a'}$, each operator of which defines one of the p^l matrices (a_{ij}) which will represent r^a. If (a_{ij}) is any particular matrix which represents r^a the others which represent it are of the form $(a_{ij}) + M(x)$, x arbitrary.

If r^a is a commutator of the ν-th kind then it must correspond to at least one commutator of the ν-kind in G^* (and to none of kind less than ν) and therefore can be represented by at least one matrix (a_{ij}) of rank ν. Now suppose that the maximum rank of the matrices representing r^a is $\nu + \nu'$. Then we shall say that *r^a is a commutator of the ν-th kind and extent ν'.*

Two groups G_1 and G_2 are isomorphic if and only if R_1 and R_2 are conjugate under some isomorphism of G^*. But in any isomorphism of G^* with itself, a commutator of the ν-th kind corresponds to a commutator of the ν-th kind, for the rank of its matrix is unchanged by new choice of generators in S, U, and K. So the number of commutators of the ν-th kind in G^* corresponding to identity in G is an invariant of G.

If two commutators r_1, r_2 in G give rise to non-isomorphic quotient groups G/r_1 and G/r_2, we are justified in distinguishing between r_1 and r_2 in G. The following theorem gives such justification to the definition of kind and extent.

THEOREM. *If in a group G two commutators*[10] *r_1 and r_2 are such that G/r_1 and G/r_2 are isomorphic, then r_1 and r_2 are of the same kind and extent.*

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[8] An obvious generalization to all groups G with $l = mk$ is: *A necessary and sufficient condition that an operator in the commutator subgroup be of the ν-th kind is that its matrix shall be of rank ν.*

[9] It is of some interest to contrast this result with early conjectures with respect to the existence of non-commutators in the commutator subgroup. See W. B. Fite, *Transactions of the American Mathematical Society*, vol. 3 (1902), pp. 331-353, in particular pp. 332 and 339.

[10] For simplicity we shall call any operator in K a commutator, recalling, of course, that it is not a commutator in the ordinary sense if its kind is greater than two.

Let r_i be of kind ν_i and extent ν'_i. Let identity in G/r_i correspond to $\{R, r'_i\}$ in G^*, where r'_i is of kind ν_i and $G^*/R = G$. Then if $\nu_1 \neq \nu_2$, say $\nu_1 < \nu_2$, $\{R, r'\}$ has more operators of kind ν_1 than $\{R, r'_2\}$, for as a consequence of the definition of kind no operator in $\{R, r'_2\}$ can be of kind less than ν_2 unless it is in R. Hence $\nu_1 = \nu_2$. Next suppose $\nu_1 = \nu_2$, but $\nu'_1 \neq \nu'_2$, say $\nu'_1 < \nu'_2$. Then $\{R, r'_2\}$ contains an operator of kind $\nu_1 + \nu'_2$ whereas no operator in $\{R, r'_1\}$ is of kind greater than $\nu_1 + \nu'_1$. Hence, the isomorphism of G/r_1 and G/r_2 implies $\nu_1 = \nu_2$ and $\nu'_1 = \nu'_2$.

The converse of this theorem is not true as is illustrated by the groups represented by the forms $M_{10}(x)$ and $M_9(x)$ (§ 7). These are both quotient groups of M_2 (§ 6) with respect to commutators of the second kind and extent zero.

6. τ-classes (1, 3, 3) and related groups. For $l = 1$, $M(x) = (a_{1ij}x_1)$ and the τ-class is completely defined by the rank of (a_{1ij}); so we have three τ-classes, represented by $M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Interpreted for $l = 1$, the first two do not define groups G. M_3 does give a group G, the only one for 1, 3, 3. Interpreted for $l = 8$ we get three groups G, say G_1, G_2, G_3, where $G_i = G^*/r^{x^{(i)}}$, $r^{x^{(i)}}$ being a commutator of the i-th kind in G^*, which gives a differentiation of the groups G according to the preceding theorem.

7. τ-classes (2, 3, 3) and related groups. For $l = 2$, $M(x) = x_1(\alpha_{ij}) + x_2(\beta_{ij})$ and a complete classification is given by the theories of invariant factors and of binary cubics. For $M(x)$ we may have: 1) an irreducible cubic, 2) a cubic with one real root, 3) a cubic with three distinct real roots, 4) a cubic with two roots, one repeated, 5) a cubic with a triple root, 6) a cubic identically zero.

It has been shown[11] that all cubics $f(x)$ belonging to and irreducible in the $GF[p]$ are conjugate under the group of linear transformations on x_1 and x_2. Hence 1) gives just one τ-class represented by $M_1(x) = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & x_2 \\ \alpha x_2 & \alpha x_2 & x_1 \end{pmatrix}$ where $x_1^3 - \alpha x_1 x_2^2 + \alpha x_2^3$ is irreducible. For 2) it is evident that we may take $f(x) = x_1(x_1^2 - \gamma x_2^2)$, $\gamma \neq$ square, giving the single τ-class represented

[11] H. R. Brahana, *Bulletin of the American Mathematical Society*, vol. 39 (1933), pp. 962-969.

by $M'_2(x) = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & x_2 \\ 0 & \gamma x_2 & x_1 \end{pmatrix}$, or if we prefer by $M_2(x) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & x_2 \\ 0 & \gamma x_2 & x_1 \end{pmatrix}$. For 3) we may take $f(x) = x_1x_2(x_1 + x_2)$ giving the τ-class represented by $M_3(x) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_1 + x_2 \end{pmatrix}$. For 4) we may take $f(x) = x_1{}^2x_2$ and have two τ-classes distinguished by their invariant factors, represented by $M_4(x) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_2 \end{pmatrix}$ and $M_5(x) = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_2 \end{pmatrix}$. For 5) we take $f(x) = x_1{}^3$ and have (since x_2 must appear) the two τ-classes represented by $M_6(x) = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 \end{pmatrix}$ and $M_7(x) = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & x_2 \\ 0 & 0 & x_1 \end{pmatrix}$. (If x_2 did not appear we should have the case 1, 3, 3 instead of 2, 3, 3).

For 6) $f(x) \equiv 0$ we list the following representatives of τ-classes with descriptive invariants sufficient to establish their distinctness:

$$M_8(x) = \begin{pmatrix} x_1 & x_2 & 0 \\ \gamma x_2 & x_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \gamma \neq \text{square}; \qquad M_9(x) = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & x_2 \\ 0 & 0 & 0 \end{pmatrix};$$

$$M_{10}(x) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & x_2 \\ x_2 & 0 & 0 \end{pmatrix}; \qquad M_{11}(x) = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$M_{12}(x) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & x_2 \\ 0 & 0 & 0 \end{pmatrix}; \qquad M_{13}(x) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$M_{14}(x) = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Every x-section of M_{14} is of rank one, which is sufficient to distinguish it from the others. M_{13} has two x-sections of rank one; M_{12} and M_{11} each one; and M_{10}, M_9, M_8 no x-sections of rank one. M_{11} and M_{12} differ in that F_{11} is free of both y_3 and z_3, whereas F_{12} is free of y_3 but not of z_3. Similarly F_8 is free of y_3 and z_3; F_9 is free of y_3; and F_{10} involves all of the variables. The completeness of this list of τ-classes for $f(x) \equiv 0$ follows from Brahana's analysis of the cases: (3, 2, 3), (2, 2, 3), (2, 2, 2), (2, 2, 1).[12]

[12] *American Journal of Mathematics*, vol. 56 (1934), pp. 490-510. He omitted the group corresponding to M_8 and listed one non-existent group $G(2, 2, 3)$.

Interpreted for $l=2$, $M_1, \cdots, M_7$ and M_{10} give the eight groups G with numbers $2, 3, 3$ and the other forms do not give groups G. Interpreted for $l=7$ we get the fourteen groups G with numbers $7, 3, 3$.

8. $l=3$; theory and method of attack. For $l=3$ in addition to the ternary cubic $|M(x)|$ we have also the ternary cubics $|M_y|$ and $|M_z|$ (where M_y and M_z are defined by the y- and z-sections of $M(a_{hij})$ just as $M(x) \equiv M_x$ is defined by the x-sections).

First, we shall classify the forms according to the projective invariants of $f(x_1, x_2, x_3) = |M(x)|$. A. D. Campbell has [13] given a projective classification of ternary cubics with coefficients in the $GF[p]$. To determine the representations of a given cubic $f(x)$ as $|M(x)|$ we shall first represent a line section of it, say $f(x_1, 0, x_3)$, as a determinant according to the methods used for the case $l=2$, giving $M(x_1, 0, x_3)$ such that $|M(x_1, 0, x_3)| = f(x_1, 0, x_3)$. Then we shall consider $M(x) = M(x_1, 0, x_3) + (a_{ij})x_2$ where the a_{ij} are to be determined so that $|M(x)| = f(x)$. For $f(x)$ we take in turn a representative from each projective class or set of classes of cubics (making several additions and corrections to Campbell's list). Aside from class 20 and the class $f(x) \equiv 0$, $f(x_1, 0, x_3)$ is one of the following binary cubics: $f_1 = x_1^3$, $f_2 = x_1^2(x_1 + x_3)$, $f_3 = x_1^2 x_3$, $f_4 = x_1(x_1^2 - x_3^2)$; and so we may suppose that $M_1(x_1, 0, x_3)$ is one of the following seven matrices:

$$M_1 = \begin{pmatrix} x_1 & x_3 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} x_1 & x_3 & 0 \\ 0 & x_1 & x_3 \\ 0 & 0 & x_1 \end{pmatrix} \text{ from } f_1;$$

$$M_3 = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 + x_3 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} x_1 & x_3 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 + x_3 \end{pmatrix} \text{ from } f_2;$$

$$M_5 = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, \qquad M_6 = \begin{pmatrix} x_1 & x_3 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \text{ from } f_3;$$

$$M_7 = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 + x_3 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \text{ from } f_4.$$

We now give the expansions of $|M_i(x_1, 0, x_3) + (a_{ij})x_2|$, $i = 1, \cdots, 7$, and determine the conditions on (a_{ij}) for

$$|M(x)| = f(x) = a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_4 x_1^2 x_2 + a_5 x_1 x_2^2 + a_6 x_1^2 x_3 + a_7 x_1 x_3^2 + a_8 x_2^2 x_3 + a_9 x_2 x_3^2 + a_{10} x_1 x_2 x_3,$$

[13] *Messenger of Mathematics*, vol. 58 (1928), pp. 33-48.

where the a_ν are given.

$|M_1(x_1, 0, x_3) + (a_{ij})x_2| = f(x)$, $a_1 = 1$, $a_3 = a_6 = a_7 = a_9 = 0$, gives B_1:

1) $a_{11} + a_{22} + a_{33} = a_4$ 3) $|a_{ij}| = a_2$

2) $a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32} = a_5$

4) $a_{23}a_{31} - a_{21}a_{33} = a_8$ 5) $-a_{21} = a_{10}$

$|M_2(x_1, 0, x_3) + (a_{ij})x_2| = f(x)$, $a_1 = 1$, $a_3 = a_6 = a_7 = 0$, gives B_2: 1), 2), 3) same as in B_1.

4) $-a_{21}a_{33} + a_{23}a_{31} + a_{12}a_{31} - a_{11}a_{32} = a_8$

5) $a_{31} = a_9$ 6) $-a_{31} - a_{32} = a_{10}$

$|M_3(x_1, 0, x_3) + (a_{ij})x_2| = f(x)$, $a_1 = a_6 = 1$, $a_3 = a_7 = a_9 = 0$, gives B_3: 1), 2), 3) same as in B_1.

4) $a_{11}a_{22} - a_{21}a_{12} = a_8$ 5) $a_{11} + a_{22} = a_{10}$

$|M_4(x_1, 0, x_3) + (a_{ij})x_2| = f(x)$, $a_1 = a_6 = 1$, $a_3 = a_7 = 0$, gives B_4: 1), 2), 3) same as in B_1.

4) $a_{11}a_{22} - a_{12}a_{21} + a_{23}a_{31} - a_{21}a_{33} = a_8$

5) $-a_{21} = a_9$ 6) $a_{11} + a_{22} - a_{21} = a_{10}$

$|M_5(x_1, 0, x_3) + (a_{ij})x_2| = f(x)$, $a_6 = 1$, $a_1 = a_3 = a_7 = a_9 = 0$, gives B_5:

1) $a_{33} = a_4$ 3) $|a_{ij}| = a_2$

2) $a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32} - a_{13}a_{31} = a_5$

4) $a_{11}a_{22} - a_{12}a_{21} = a_8$ 5) $a_{11} + a_{22} = a_{10}$

$|M_6(x_1, 0, x_3) + (a_{ij})x_2| = f(x)$, $a_6 = 1$, $a_1 = a_3 = a_7 = 0$, gives B_6: 1), 2), 3) same as in B_5.

4) $a_{11}a_{22} + a_{23}a_{31} - a_{12}a_{21} - a_{21}a_{33} = a_8$

5) $-a_{11} = a_9$ 6) $a_{33} - a_{22} = a_{10}$

$|M_7(x_1, 0, x_3) + (a_{ij})x_2| = f(x)$, $a_1 = a_7 = 1$, $a_3 = a_6 = 0$, gives B_7: 1), 2), 3), same as in B_1.

4) $-a_{11}a_{22} + a_{11}a_{33} + a_{12}a_{21} - a_{13}a_{31} = a_8$

5) $-a_{11} = a_9$ 6) $a_{33} - a_{22} = a_{10}$.

Two solutions $M(x)$ and $M'(x)$ of $|M(x)| = f(x)$, such that $M(x_1, 0\ x_3) = M'(x_1, 0, x_3)$, belong to the same τ-class if $M' = PMQ$, which implies $PM(x_1, 0, x_3)Q = M(x_1, 0, x_3)$. We must therefore determine the pairs of matrices P and Q such that $PM_i(x_1, 0, x_3)Q = M_i(x_1, 0, x_3)$ for $i = 1, \cdots, 7$. To show that $PM_i(x_1, 0, x_3)Q = M_i(x_1, 0, x_3)$ identically in x_1 and x_3, it is sufficient to show that it is true for any independent pair of number couples $(x_1, 0, x_3)$. Proceeding thus we find that in each case $Q = P^{-1}$. Let P_i be the most general constant non-singular matrix permutable with M_i. Then a simple computation (which we shall omit) gives:

$$P_1=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ 0 & \alpha_{11} & 0\\ 0 & \alpha_{32} & \alpha_{33}\end{pmatrix};\quad P_2=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ 0 & \alpha_{11} & \alpha_{12}\\ 0 & 0 & \alpha_{11}\end{pmatrix};\quad P_3=\begin{pmatrix}\alpha_{11} & \alpha_{12} & 0\\ \alpha_{21} & \alpha_{22} & 0\\ 0 & 0 & \alpha_{33}\end{pmatrix};$$

$$P_4=\begin{pmatrix}\alpha_{11} & \alpha_{12} & 0\\ 0 & \alpha_{11} & 0\\ 0 & 0 & \alpha_{33}\end{pmatrix};\quad P_5=\begin{pmatrix}\alpha_{11} & \alpha_{12} & 0\\ \alpha_{21} & \alpha_{22} & 0\\ 0 & 0 & \alpha_{33}\end{pmatrix};\quad P_6=\begin{pmatrix}\alpha_{11} & \alpha_{12} & 0\\ 0 & \alpha_{11} & 0\\ 0 & 0 & \alpha_{33}\end{pmatrix};$$

$$P_7=\begin{pmatrix}\alpha_{11} & 0 & 0\\ 0 & \alpha_{22} & 0\\ 0 & 0 & \alpha_{33}\end{pmatrix};$$

the elements in each matrix being arbitrary, except for the negative restriction that none of the matrices be singular.

The matrices P_1 form a group (under matrix multiplication) generated by

$$T_{12}(m)=\begin{pmatrix}1 & m & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad T_{13}(m)=\begin{pmatrix}1 & 0 & m\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad T_{32}(m)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & m & 1\end{pmatrix},$$

$$T_{33}(m)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & m\end{pmatrix},\quad T(m)=\begin{pmatrix}m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & m\end{pmatrix},$$

m arbitrary in each generator. Transformation of a matrix (α_{ij}) by $T_{ij}(m)$ effects 1) adding m times the i-th column to the j-th column and 2) adding $-m$ times the j-th row to the i-th row. Let $t_{ij}(m)$ represent this operation of transformation. Transformation of (α_{ij}) by $T_{ii}(1/m)$ effects: 1) multiplying the i-th row by m and 2) multiplying the i-th column by $1/m$. Let $t_{ii}(m)$ represent this operation of transformation. Transformation by $T(m)$ effects no change. Then the group $T_1=\{t_{12}(m), t_{13}(m), t_{32}(m), t_{33}(m)\}$ (m arbitrary in each generator) completely expresses the transformations of (α_{ij}) by the group P_1. Defining $T_2,\cdots,T_7$ in the same manner, we have

$$T_2=\{t_{12}(m)\cdot t_{23}(m), t_{13}(m)\};\qquad T_3=\{t_{11}(m), t_{22}(m), t_{12}(m), t_{21}(m)\};$$
$$T_4=\{t_{33}(m), t_{12}(m)\};\quad T_5=T_3;\quad T_6=T_4;\quad T_7=\{t_{11}(m), t_{22}(m)\}.$$

We separate the matrices $M(x)$ having a given determinant and a given x-section $M_i(x_1, 0, x_3)$ into sets such that the members of any given set include all the matrices equivalent to any single member under the group T_i just defined, and τ. Then we shall consider the automorphisms of $f(x)$. Let $x\to x'$ be an automorphism of $f(x)$. We have then $f(x)=g(x')=mf(x')$ and hence $f(x_1,0,x_3)=mf(x'_1,0,x'_3)$. Let $M(x_1,x_2,x_3)\to M'(x'_1,x'_2,x'_3)$. Then $|M'(x_1,0,x_3)|=mf(x_1,0,x_3)$. Now $M'(x_1,0,x_3)$ may not be one of

the canonical x-sections $M_i(x_1, 0, x_3)$. If it is not we can find constant matrices P and Q such that $M''(x) = PM'(x)Q$ where $M''(x_1, 0, x_3)$ is one of the canonical x-sections. We say that $M(x)$ is replaced by $M''(x)$ under the given automorphism. The sets of solutions including $M(x)$ and $M''(x)$ are in the same τ-class, and, conversely, if two of the above sets of solutions are such that for no automorphism is any member of one replaced by any member of the other, then the two sets are in distinct τ-classes.

Summarizing, to determine the complete set of τ-classes for $l = 3$, $m = k = 3$: 1) select a representative from each projective class of ternary cubics and determine its automorphisms; 2) determine a canonical set of x-sections $M_i(x_1, 0, x_3)$ such that $|M_i(x_1, 0, x_3)| = f(x_1, 0, x_3)$ and the pairs of constant matrices P and Q such that $PM_i(x_1, 0, x_3)Q = M_i(x_1, 0, x_3)$; 3) determine the matrices (a_{ij}) such that $|M_i(x_1, 0, x_3) + (a_{ij})x_2| = f(x)$, and separate these into sets whose members are equivalent under the pairs P, Q, and τ; 4) combine such of these sets as are equivalent under the automorphisms of $f(x)$, this final combination giving the τ-classes 3, 3, 3 (and 6, 3, 3). We shall insist throughout that no x-section be of rank zero as that implies $l < 3$.[14]

9. $l = 3$; actual determination of τ-classes (3, 3, 3). We shall now list the τ-classes (3, 3, 3) for each class or set of classes of cubics, giving the complete computation in certain typical cases, and in others merely indicating the procedure.

Class 1) $f(x) = x_1^3 - x_2^2 x_3$. The group of automorphisms, A_1, is generated by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}$. Here we use M_1 and M_2. With M_1 we have equations B_1 with $a_4 = a_5 = a_2 = a_{10} = 0$, $a_8 = -1$. Equation 5) gives $a_{21} = 0$ and this in 4) gives $a_{23}a_{31} = -1$ whence $a_{23}a_{31} \neq 0$. Now supposing that we had a solution of equations B_1 with $a_{32} \neq 0$, we would have after using $t_{12}(-a_{32}/a_{31})$ an equivalent solution with $a'_{32} = 0$. For

$$(a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

[14] There is an obvious generalization of this paragraph for the cases $l = 3$, $m = k$ for all values of m. But for $m > 3$ the difficulties of computation become very great, especially for the first of the four steps indicated. The methods indicated here will work not only for the $GF[p]$, but for trilinear forms in any field. If, however, one is working in an algebraically closed field somewhat simpler methods will apply. The author is now working on the 3, 3, 3 case for the complex number field.

becomes

$$(a'_{ij}) = \begin{pmatrix} a_{11} & a_{12} + (a_{22} - a_{11})(a_{32}/a_{31}) & a_{13} + (a_{23}a_{32}/a_{31}) \\ 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}.$$

Hence we lose no generality in supposing $a_{32} = 0$ in equations B_1. Similarly using $t_{13}(-a_{33}/a_{31})$ we have $a'_{33} = 0$. Then applying $t_{33}(1/a_{31})$ we have, in view of 4), $a'_{31} = 1 - a'_{23}$. Then $t_{32}(a_{22}) \cdot t_{13}(-a_{22})$ gives $a'_{22} = a'_{32} = a'_{33} = a'_{21} = 0$. Substituting in B_1 we have:

1) $a_{11} = 0$; 2) $-a_{13}a_{31} = 0$ whence $a_{13} = 0$; 3) $-a_{12} = 0$

giving $(a_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$ and representing the only τ-class for

$$M(x_1, 0, x_3) = M_1.$$

Now from M_2 we get equations B_2 with $a_4 = a_5 = a_2 = a_9 = a_{10} = 0$, $a_8 = -1$. Substituting 5) $a_{31} = 0$ and 6) $-a_{21} - a_{32} = 0$ in 4), we have $a_{32}(a_{33} - a_{11}) = -1$. Hence $a_{32} \neq 0$. Operating with

$$t_{12}(-a_{33}/a_{32}) \cdot t_{23}(-a_{33}/a_{32})$$

we have $a'_{33} = 0$, $a'_{11} \neq 0$. Now $t_{13}(-a_{13}/a_{11})$ gives $a'_{13} = 0$ and B_2 becomes:

1) $a_{11} + a_{22} = 0$ or $a_{22} = -a_{11}$; 2) $a_{11}a_{22} - a_{23}a_{32} + a_{12}a_{32} = 0$;
3) $-a_{11}a_{23}a_{32} = 0$, hence $a_{23} = 0$; 4) $a_{11}a_{32} = 1$;

giving

$$(a_{ij}) = m(a) = \begin{pmatrix} 1/a & 1/a^3 & 0 \\ -a & 1/a & 0 \\ 0 & a & 0 \end{pmatrix}; \quad a \neq 0.$$

Now we use the automorphisms of $f(x)$, and τ to investigate the possibility of $m(a) \overset{\tau}{\sim} m(a')$. The automorphism $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}$, $\alpha \neq 0$, gives $m(a) \sim m(\alpha a)$ and therefore the single τ-class given by $m(1) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Classes 2) and 3) $f(x) = x_1^3 + x_1^2x_2 - \gamma x_3^2x_2$. For class 2) $\gamma = 1$,

$$A_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} c & 0 & 1 \\ -9c/4 & -2c & -3/4 \\ -1 & 0 & c \end{pmatrix} \text{ where } c^2 = -1/3 \right\};$$

and for class 3), $\gamma \neq$ square,

$$A_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 12 \\ -9 & -8 & -9 \\ 4 & 0 & -4 \end{pmatrix} \text{ where } p = 6k+5, \gamma = -3 \right\}.$$

$f(x_1, 0, x_3) = x_1^3$ and since there are terms in x_3^2 in $f(x)$ we must have $M(0,0,1)$ of rank two; hence we need only consider M_2 giving B_2 with $a_4 = 1$, $a_9 = -\gamma$, $a_5 = a_2 = a_8 = a_{10} = 0$. There are for each of these two classes of cubics $(p+3)/2$ τ-classes represented by $m_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ -a & 1 & 0 \\ -\gamma & a & 0 \end{pmatrix}$, $m(a) \overset{\tau}{\sim} m(-a)$; and $m_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -\gamma & 0 & 0 \end{pmatrix}$.

Class 4) $f(x) = x_1^3 + \alpha x_2^3 + x_1 x_2 x_3$, $\alpha \neq$ cube, whence $p = 6k+1$. Campbell[15] lists this as set 4) giving a cubic with parameter α as equivalent to another with parameter $\alpha' = k^3\alpha$ for any k. He failed to consider interchanging the two tangents at the double point, which gives cubics with parameters α and α' in the same class if $\alpha' = k^3\alpha^2$. But if $\alpha \neq$ cube, any other not-cube in the $GF[p]$ is one of the forms $k^3\alpha$ or $k^3\alpha^2$ for some k. A_4 is generated by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$, $\omega^3 = 1$. x_3^2 does not appear in $f(x)$; so we have cases with both M_1 and M_2. From M_1 we get the single τ-class represented by $m_1 = \begin{pmatrix} 0 & 0 & -\alpha \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and from M_2 the $(p-1)/2$ τ-classes represented by $m_2(a) = \begin{pmatrix} 0 & 0 & -\alpha/a(1+a) \\ -1-a & 0 & 0 \\ 0 & a & 0 \end{pmatrix}$, $a \neq 0, -1$; $m_2(a) \overset{\tau}{\sim} m_2(-a)$.

Class 5) $f(x) = x_1^3 + \alpha x_2^3 + x_1^2 x_3 + 3x_2^2 x_3$, $p = 6k+5$. A is generated by $\begin{pmatrix} 1 & 3 & 0 \\ -1 & 1 & 0 \\ (\alpha-9)/4 & (-3\alpha-9)/4 & -2 \end{pmatrix}$. Campbell's result differs somewhat from this due to a mistake in calculation. On p. 35[16] the flex equation should be $\gamma\alpha c^3 - 3c^2 + 3\gamma^2\alpha c - \gamma = 0$ instead of $\gamma\alpha c^3 + 3c^2 - 3\gamma^2\alpha c - \gamma = 0$ as it is there. Then his conditions on α and p that the set exist become $-(3/\gamma) =$ square or since $\gamma \neq$ square; $-3 \neq$ square, $p = 6k+5$; and

[15] *Loc. cit.*, p. 35.

[16] *Loc. cit.*

$\alpha \neq \beta\left(\dfrac{\beta^2-1}{9\beta^2-1}\right)$ for any β in the $GF[p]$. (Since γ is any particular not-square we take $\gamma = -3$.) Further $1 + \alpha\gamma^{3/2}$, $\alpha \neq 0$, is a cube in the $GF[p^2]$ for only $(p-5)/3$ values of α. The other $(2p+2)/3$ will not give cubes and are therefore not of the form $\beta\left(\dfrac{\beta^2-1}{9\beta^2-1}\right)$.

The reducibility of $g(x, \alpha'') = \gamma x^3 + 3\gamma^2\alpha'' x + 3x + \gamma\alpha''$, $\alpha'' = \dfrac{\alpha' - \alpha}{\alpha\alpha'\gamma^3 - 1}$, is the condition that $f(x, \alpha)$ be equivalent to $f(x, \pm\alpha')$ i. e. if using either one of α' and $-\alpha'$ gives the cubic $g(x, \alpha'')$ reducible, then $f(x, \alpha)$ is equivalent to both $f(x, \alpha')$ and $f(x, -\alpha')$. But the reducibility of $g(x, \alpha'')$ implies $\alpha'' = \beta\left(\dfrac{\beta^2-1}{9\beta^2-1}\right)$ for some β. That is, α'' is one of the set B of $(p+1)/3$ numbers (including 0 and ∞) giving cubics $f(x, \alpha'')$ with flexes; conversely, if α'' is any such number, from $\alpha'' = \dfrac{\alpha' - \alpha}{\alpha\alpha'\gamma^3 - 1}$ we have $\alpha' = \dfrac{\alpha'' - \alpha}{\alpha\alpha''\gamma^3 - 1}$ where $f(x, \alpha')$ is equivalent to $f(x, \alpha)$. As α'' takes its $(p+1)/3$ possible values, α' takes $(p+1)/3$ distinct values (including α). Now if $-\alpha'$ is not in this set for each (or any) α' we have $(2p+2)/3$ distinct numbers giving equivalent cubics $f(x, \pm\alpha')$. But the set A of numbers α giving cubics $f(x, \alpha)$ without flexes contains just $(2p+2)/3$ numbers. Hence, in this case all of the cubics with acnodes but no (real) flex points belong to one projective class. We shall show that $-\alpha'$ and α' are never in the same set defined by $\dfrac{\alpha'' - \alpha}{\alpha\alpha''\gamma^3 - 1}$ for fixed α in A, α'' varying, as above, in B and hence that there is always just one class.

For suppose that $\alpha' = \dfrac{\alpha'' - \alpha}{\alpha\alpha''\gamma^3 - 1}$ and $-\alpha = \dfrac{\beta'' - \alpha}{\alpha\beta''\gamma^3 - 1}$, α'' and β'' in B. Then

$$\beta'' = \frac{\alpha'' - \left(\dfrac{2\alpha}{1 + \alpha^2\gamma^3}\right)}{\left(\dfrac{2\alpha}{1 + \alpha^2\gamma^3}\right) - 1}$$

which implies that $2\alpha/(1 + \alpha^2\gamma^3)$ is in B. Further since α can be taken as any one of the $(2p+2)/3$ numbers in the set A, $2\alpha/(1 + \alpha^2\gamma^3)$ must be in B for every α in A. But $a = 2\alpha/(1 + \alpha^2\gamma^3)$ cannot have more than two solutions α for any a. Hence as α takes every value in A, $2\alpha/(1 + \alpha^2\gamma^3)$ must take at least $(p+1)/3$ distinct values, and being always in B must fill it. But since 0 is in B this implies $0 = 2\alpha/(1 + \alpha^2\gamma^3)$, or $\alpha = 0$ or $\alpha = \infty$ for α in A. But $0, \infty$ are not in A, giving a contradiction to the supposition that

α' and $-\alpha'$ had the above representations, which completes the proof that all such cubics are projectively equivalent. $f(x_1, 0, x_3) = x_1^2(x_1 + x_3)$ so we need to consider both M_3 and M_4. From M_3 we have the single solution $m_1 = \begin{pmatrix} 0 & 1 & 3 \\ -3 & 0 & \alpha \\ 1 & 0 & 0 \end{pmatrix}$, and from M_4 the $(p+1)/2$ τ-classes given by

$$m_2(a) = \begin{bmatrix} -a & \dfrac{\alpha - a^3}{3 + a^2} & -a^2 \\ 0 & a & 3 + a^2 \\ 1 & 0 & 0 \end{bmatrix}, \quad m_2(a) \overset{T}{\sim} m_2(-a).$$

Sets 6) and 7) $f(x) = x_1(x_1 - x_2)(x_1 - \alpha x_2) - \gamma x_2 x_3^2$; $\gamma = 1$ (set 6), $\gamma \neq$ square (set 7). $A_6 = A_7 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sqrt{-1} \end{pmatrix} \text{ if } \alpha = 1/2, \begin{pmatrix} 1 & \omega & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} (\omega^3 = 1) \text{ if } \alpha^2 - \alpha + 1 = 0 \right\}$. Here we need only consider M_2 giving the solutions $m(a, b) = \begin{pmatrix} -1-\alpha-a & b & -b^2/a \\ 0 & a & -b \\ -\gamma & 0 & 0 \end{pmatrix}$ where $b = (-a/\gamma)(\alpha + a)(1 + a)$ and of course values of a for which b would not be real are discarded. For all values of α, $m(a, b) \overset{T}{\sim} m(a, -b)$; if $\alpha = 1/2$ and $p = 4k + 1$, $m(a, b) \overset{T}{\sim} m(-1 - a, b)$. If $\alpha^2 - \alpha + 1 = 0$, $m(a, b) \overset{T}{\sim} m(\omega a + \omega, b)$. Aside from these cases any two values of a for which b is real give distinct τ-classes.

Sets 8) and 9) $f(x) = x_1(x_1^2 - \alpha x_1 x_2 + \alpha x_2^2) - \gamma x_2 x_3^2$, $\alpha^2 - 4\alpha \neq$ square, $\gamma = 1$ (set 8), $\gamma \neq$ square (set 9). $A_8 = A_9 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$. We have only M_2 to consider giving the solutions $m(a, b) = \begin{pmatrix} -\alpha - a & b & -b^2/a \\ 0 & a & -b \\ -\gamma & 0 & 0 \end{pmatrix}$, $\gamma b^2 = -a(a^2 + \alpha a + \alpha)$, $m(a, b) \overset{T}{\sim} m(a, -b)$. There is a τ-class for each a giving b real.

Set 8a) (two classes) $f(x) = x_1(x_1^2 - \alpha x_2^2) - x_2 x_3^2$, $\alpha = \gamma$ or $1/\gamma$, $\gamma \neq$ square. (Campbell omits this set.) $A_{8a} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sqrt{-1} \end{pmatrix} \right\}$.

The τ-classes are given by $m(a, b) = \begin{pmatrix} -a & b & a^2 - \alpha \\ 0 & a & b \\ -1 & 0 & 0 \end{pmatrix}$, $b^2 = -a(a^2 - \alpha)$,

with $m(a,b) \overset{T}{\sim} m(a,-b)$ and if $-1 =$ square $m(a,b) \overset{T}{\sim} m(-a, \sqrt{-1}\, b)$ Otherwise values of a for which b is real give distinct τ-classes.

Sets 10) and 11) $f(x) = x_1^3 - \alpha x_1 x_2^2 + \alpha x_2^3 - \gamma x_2 x_3^2$, $\gamma = 1$ (set 10), $\gamma \neq$ square (set 11). $A_{10} = A_{11} = A_8$. The τ-classes are given by $m(a,b) = \begin{pmatrix} -a & b & (a^2-\alpha)/\gamma \\ 0 & a & -b \\ -\gamma & 0 & 0 \end{pmatrix}$, $\gamma b^2 = a^3 - \alpha a + \alpha$, $m(a,b) \overset{T}{\sim} m(a,-b)$.

Set 12) $f(x) = x_1^3 - \alpha x_2^3 - x_2 x_3^2$, $\alpha \neq$ cube.

$$A_{12} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\omega^3 = 1) \right\}.$$

The τ-classes are given by $m(a,b) = \begin{pmatrix} -a & b & a^2 \\ 0 & a & -b \\ -1 & 0 & 0 \end{pmatrix}$, $b^2 = \alpha - a^3$, where $m(a,b) \overset{T}{\sim} m(\omega a, -b)$.

Sets 13) and 16) $f(x) = x_1^3 - x_2 x_3^2 + \alpha x_2^2 x_3$.

$$A_{13} = \left\{ \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\omega^3 = 1), \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -\alpha^2 & 0 \end{pmatrix}, \begin{pmatrix} -\alpha^2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -\alpha^2 & \alpha \end{pmatrix} \right\}.$$

A_{16} same as A_{13} except that there may be additional automorphisms obtained by taking for $x'_1 = 0$ any other line of flexes. (Such transformation will give either automorphisms or equivalent cubics in the same form but with different values of α). The τ-classes are given by $m(a,b) = \begin{pmatrix} -a & b & a^2 \\ 0 & a & -b-\alpha \\ 1 & 0 & 0 \end{pmatrix}$, $b^2 + \alpha b + a^3 = 0$, where the two roots b for a given a give equivalent solutions, and $m(a,b) \sim m(\omega a, b)$.

Sets 14) and 15) $f(x) = x_1^3 + x_2 x_3 (x_1 - \alpha x_2 - x_3)$.

$$A_{14} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/\alpha \\ 0 & \alpha & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1/\alpha & -1 & -(1/\alpha) \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

$A_{15} = A_{14}$ plus automorphisms that might arise from taking for $x'_1 = 0$ another of the twelve flex lines. Campbell's discussion as to values of α giving set 14), i. e. just three real flex points, is incomplete. He gives [17] sufficient but not necessary conditions on α, and makes a numerical error in the calculations that he does give. He solves $x = ay + bz$ simultaneously with

[17] *Loc. cit.*, p. 43.

$f(x, y, z) = \alpha z^3 + xy(z - x - y) = 0$ and requires that the result be a perfect cube. This gives a rationally in terms of b and c where $b^2 - b + 3\alpha c = 0$ and c is a root of $3\alpha c^3 - c^2 - 3c - 3 = 0$. The discriminant of this cubic is a square multiple of -3. Now if a and b are real the curve $f(x, y, z) = 0$ has four, and therefore nine, real flex points. But then there must be six distinct real solutions b, which implies that the cubic in c has three real zeros. But the cubic has three real zeros only when $-3 =$ square and α is of the form $(c^2 + 3c + 3)/3c^3$.[18] Otherwise, i. e. $-3 \neq$ square or $-3 =$ square, $\alpha \neq (c^2 + 3c + 3)/3c^3$, the curve $f(x, y, z)$ has just three real flex points. The representation given above is readily derived from this one. The τ-classes are given by $m(a, b) = \begin{pmatrix} -a & b & a^2 - b \\ -1 & a & \alpha - b \\ -1 & 0 & 0 \end{pmatrix}$, $b^2 - (a + \alpha)b + a^3 = 0$, the two roots b for a given a giving equivalent τ-classes.

Sets 17), 18), 19) $f(x) = x_1{}^2x_3 + x_1x_2{}^2 + \alpha x_2x_3{}^2 + ax_1{}^2x_2 + bx_1x_2x_3$, $a = 1$, $b = -\beta$ (set 17); $a = 0$, $b = 1$ (set 18); $a = b = 0$ (set 19). For sets 17), 18), and 19) the computations involved in determining the classes and then the complete group of automorphisms seem too involved to be worth doing. For sets 18) and 19) we list the automorphism $\begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \alpha \\ 1 & 0 & 0 \end{pmatrix}$ and for set 19) the further automorphism $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$. The τ-classes for set 17) are given by $m_1(a, b) = \begin{pmatrix} -\beta & -b/\alpha - 1 & -ab - 1 - \beta \\ -\alpha & 0 & b \\ 1 & a & 1 \end{pmatrix}$, $(\alpha b)a^2 + (\alpha + \alpha\beta + \beta b)a - (1/\alpha)(\alpha + b)^2 = 0$, the two roots a for given b giving equivalent solutions. For set 18) $m_2(a, b) = \begin{pmatrix} 1 & -b/\alpha & -ab - 1 \\ -\alpha & 0 & b \\ 1 & a & 0 \end{pmatrix}$, $(\alpha b)a^2 + (\alpha - b)a - b^2/\alpha = 0$, the two roots a for given b giving equivalent solutions. For set 19) $m_3(a, b) = \begin{pmatrix} 0 & -b/\alpha & -ab - 1 \\ -\alpha & 0 & b \\ 1 & a & 0 \end{pmatrix}$,

$$(\alpha b)a^2 + \alpha a - b^2/\alpha = 0,$$

the two roots a for given b giving equivalent solutions and also

$$m_3(a, b) \overset{T}{\sim} m_3(\omega a, \omega^2 b).$$

[18] L. E. Dickson, *Bulletin of the American Mathematical Society*, vol. 13 (1906), pp. 1-8.

Class 20) A cubic without real points. L. E. Dickson[19] proved the existence of such cubics showing that they were all conjugate under the collineation group and that each such cubic consists of three imaginary lines. We give here another proof of the existence of cubics of this class.

Consider any binary cubic $F(x) = x_1^3 - a_1x_1^2x_2 - a_2x_1x_2^2 - a_3x_2^3$ belonging to and irreducible in the $GF[p]$. The matrix $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3 & a_2 & a_1 \end{pmatrix}$ has for its characteristic equation $-F(x,1) = 0$. *The matrices in the* $GF[p]$ *permutable with* T *are all powers of some matrix whose characteristic determinant is a primitive irreducible cubic, and of which* T *is a power.*[20] This can be seen by transforming T into the irrational canonical form $T^* = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^p & 0 \\ 0 & 0 & \rho^{p^2} \end{pmatrix}$ where ρ is a root of $F(x,1) = 0$. If σ is a primitive mark in the $GF[p^3]$ we have $\rho = \sigma^n$. Then $T^* = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma^p & 0 \\ 0 & 0 & \sigma^{p^2} \end{pmatrix}^n$ and is therefore permutable with the powers of $T^*_\sigma = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma^p & 0 \\ 0 & 0 & \sigma^{p^2} \end{pmatrix}$. There exists a matrix S such that $T_\sigma = S^{-1}T^*_\sigma S$ is in the $GF[p]$. Then $S^{-1}T^*S$ is likewise in the $GF[p]$ and has for its rational canonical form, T, and so there exists S' such that $S'^{-1}(S^{-1}T^*S)S' = T$. Then the $p^3 - 1$ distinct powers of $S'^{-1}T_\sigma S'$ are permutable with T. We complete the proof by showing that there are just $p^3 - 1$ matrices permutable with T and will further exhibit their form. If $(\alpha_{ij})T = T(\alpha_{ij})$ we have

$$\begin{pmatrix} a_3\alpha_{13} & \alpha_{11} + a_2\alpha_{13} & \alpha_{12} + a_1\alpha_{13} \\ a_3\alpha_{23} & \alpha_{21} + a_2\alpha_{23} & \alpha_{22} + a_1\alpha_{23} \\ a_3\alpha_{33} & \alpha_{31} + a_2\alpha_{33} & \alpha_{32} + a_1\alpha_{33} \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ a_3\alpha_{11} + a_2\alpha_{21} + a_1\alpha_{31} & a_3\alpha_{12} + a_2\alpha_{22} + a_1\alpha_{32} & a_3\alpha_{13} + a_2\alpha_{23} + a_1\alpha_{33} \end{pmatrix}$$

giving $(\alpha_{ij}) = \alpha_{11}I + \alpha_{12}T + \alpha_{13}T^2$, where the α_{1j}'s are arbitrary (in the $GF[p]$). There are just $p^3 - 1$ such matrices and they are therefore the $p^3 - 1$ powers of the matrix $S'^{-1}T_\sigma S'$.

Now consider

[19] L. E. Dickson, *Bulletin of the American Mathematical Society*, vol. 14, ser. 2 (1908), pp. 160-169.

[20] Compare with Jordan, *Traité de Substitutions* (1870), 128 ff.

$$f(x) = | x_1 I + x_2 T + x_3 T^2 | = x_1^3 + a_1 x_1^2 x_2 - a_2 x_1 x_2^2 + a_3 x_2^3$$
$$+ a_1 a_3 x_2^2 x_3 - a_2 a_3 x_2 x_3^2 + a_3^2 x_3^3 + (2a_2 + a_1^2) x_1^2 x_3$$
$$+ (a_2^2 - a_1 a_3) x_1 x_3^2 + (-3a_3 - a_1 a_2) x_1 x_2 x_3.$$

$f(x_1, x_2, x_3) = 0$ implies $x_1 = x_2 = x_3 = 0$, for otherwise some power of a non-singular matrix would have a zero determinant.

Now suppose that $F(x)$ is primitive. Then $M(x) = x_1 I + x_2 T + x_3 T^2$ gives a form whose determinant is the imaginary cubic. We now search for the general solution of $| x_1 I + x_2 T + x_3 (a_{ij}) | = f(x)$. We get the following equations, B_8:

1) $a_{11} + a_{22} + a_{33} = 2a_2 + a_1^2$ 3) $| a_{ij} | = a_3^2$

2) $a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32} = a_2^2 - a_1 a_3$

4) $- a_{21}a_{33} + a_{23}a_{31} + a_{12}a_{31} - a_{11}a_{32} + a_3 a_{12}a_{23} - a_3 a_{13}a_{22} + a_2 a_{13}a_{21}$
$- a_2 a_{11}a_{23} + a_1 a_{11}a_{22} - a_1 a_{12}a_{21} = - a_2 a_3$

5) $a_{11} + a_3 a_{12} + a_3 a_{23} - a_2 a_{11} - a_1 a_{21} = a_1 a_3$

6) $a_1 a_{22} - a_2 a_{23} - a_{32} - a_{21} - a_3 a_{13} = - 3a_3 - a_1 a_2.$

Now eliminating a_{33}, a_{32}, a_{31} from 2), 3), and 4) by means of 1), 5), and 6) we have three conditions remaining, two quadratic and one cubic, on the six elements of the first two rows of (a_{ij}). We cannot have $a_{11} = a_{12} = a_{13} = 0$. For any other of the p^3 sets of values for a_{11}, a_{12} and a_{13} we have at most twelve solutions of 2), 3), and 4) in a_{21}, a_{22}, and a_{23} and hence at most $12(p^3 - 1)$ solutions for the system of equations.

Now let $A = (a_{ij})$ represent any solution of B_8 not a power of T. Then $T^{-\alpha}(x_1 I + x_2 T + x_3 A) T^{\alpha}$ gives an equivalent solution $(x_1 I + x_2 T + x_3 T^{-\alpha} A T^{\alpha})$. Since A is not a power of T and since A belongs to the exponent $(p^3 - 1)/2$ (having the same characteristic polynomial as T^2), the equation $T^{-\alpha} A T^{\alpha} = A$ implies $\alpha = k(p^2 + p + 1)$. Hence the solutions A, not powers of T, can be grouped into sets of $p^2 + p + 1$, the members of any set being in the same τ-class.

Now if A is a power of T, say $A = T^{\alpha}$, since it has the same characteristic polynomial as T^2 we must have $A = T^2$, T^{2p} or T^{2p^2}, for if a matrix P has an irreducible characteristic polynomial the only ones of its powers conjugate to it (i.e. having the same characteristic polynomial) are $P^{p^{\beta}}$, $\beta = 0, 1, \cdots$.

Now it is readily verified that $T' = \begin{pmatrix} 0 & 0 & a_3 \\ 1 & 0 & a_2 \\ 0 & 1 & a_1 \end{pmatrix}$ is an automorphism of $f(x)$ replacing $f(x)$ by $a_3 f(x')$, and our solution by

$$(x'_1 T + x'_2 A + x'_3 (a_3 I + a_2 T + a_1 A)).$$

Now multiply on the right by T^{-1} giving

$$(x'_1 I + x'_2 A T^{-1} + x'_3 [a_3 T^{-1} + a_2 I + a_1 A T^{-1}]).$$

Now

$$| x_1 I + x_2 T | = | x_1 I + x_2 A T^{-1} |$$

so if A is a power of T, AT^{-1} must be conjugate to T and hence $AT^{-1} = T$, T^p or T^{p^2} giving $A = T^2$, T^{p+1} or T^{p^2+1}. This combined with the above restrictions on A gives $A = T^2$.

Hence, we have at most $\dfrac{12(p^3 - 1)}{p^2 + p + 1} = 12(p - 1)$ τ-classes with the determinant $f(x)$. Of these we have proved the existence of at least one.

It is worth mentioning here that the scheme of classification being followed gives complete results in all cases save sets 15-19 (where we listed all possible solutions but could not prove them all distinct) and class 20 (where we have been unable to list all possible solutions).

Class 21) $f(x) = x_1^3 + x_1 x_2 x_3$ gives M_1 and M_2. (We shall not list the automorphisms for the reducible cubics.) The τ-classes are:

$$m_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad m_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

from $M_1(x_1, 0, x_3)$ and

$$m_3(a) = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & -1-a & 0 \end{pmatrix}, \ a(a+1) \neq 0, \ m_3(a) \overset{\tau}{\sim} m_3(-1-a);$$

$$m_4(a) = \begin{pmatrix} 0 & 0 & a \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ a \neq 0, \ m_4(a) \sim m_4(\alpha^3 a) \sim m(\alpha^3 a^2),$$

α arbitrary, from $M_2(x_1, 0, x_3)$.

Class 22) $f(x) = x_1(x_1^2 - x_3^2 + \gamma x_2^2)$. The τ-classes (all from $M_7(x_1, 0, x_3)$) are:

$$m_1(a) = \begin{pmatrix} 0 & 1 & a \\ 0 & 0 & -\gamma \\ 0 & 1 & 0 \end{pmatrix}, \ m_1(a) \overset{\tau}{\sim} m_1(-a), \ m_1(a) \overset{\tau}{\sim} m_1(a')$$

where

$$a' = \frac{b(2c+1)a - (2c-1)(c+1)}{(c+1)(2c-1)(a/\gamma) - b(2c+1)}, \ \gamma b^2 = c^2 - 1;$$

$$m_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\gamma \\ 0 & 1 & 0 \end{pmatrix}; \quad m_3(a) = \begin{bmatrix} 0 & \dfrac{-\gamma + a^2}{2} & \dfrac{-\gamma + a^2}{2} \\ 1 & 0 & a \\ 1 & -a & 0 \end{bmatrix}, \ m_3(a) \overset{\tau}{\sim} m_3(-a).$$

The values a such that $m_1(a) \sim m_1(0)$ are given by $a = \dfrac{(2c-1)(c+1)}{b(2c+1)}$. Eliminating b from this and $\gamma b^2 = c^2 - 1$ we get the cubic equation $C)\ c^3 - \frac{3}{4}c - \frac{1}{4}\left(\dfrac{a^2+\gamma}{a^2-\gamma}\right) = 0$. For $m_1(a) \sim m_1(0)$ it is necessary and sufficient that $C)$ have a root c such that b is real. The discriminant of $C)$ is $(-3\gamma)\left[\dfrac{2(a^2-\gamma)}{3a}\right]^2$. If it is a not-square i. e. if $-3 =$ square $(p = 6k+1)$ $C)$ will have one and just one real root, c, for every $a \neq 0$, a and $-a$ giving the same root.[21] Then the values of a for which b is real and different from zero [22] will give $p-1$ values $a \neq 0$ for which $m_1(a) \sim m_1(0)$ and therefore just one τ-class for $p = 6k+1$.

If $m_1(a) \sim m_1(a')$ for more than one value of c we have

$$\frac{b(2c+1)a - (2c-1)(c+1)}{(c+1)(2c-1)(a/\gamma) - b(2c+1)} = \frac{b'(2c'+1)a - (2c'-1)(c'+1)}{(c'+1)(2c'-1)(a/\gamma) - b'(2c'+1)}$$

which implies

$$\frac{(2c-1)(c+1)}{b(2c+1)} = \frac{(2c'-1)(c'+1)}{b'(2c'+1)}$$

or in other words that $C)$ shall have two and therefore three real roots. Furthermore it is evident that if c, b, and c' are real, then b' must also be real, and hence that if $C)$ has three real roots either none or all of them give b real. Thus for $-3 \neq$ square the $(p-1)/2 - 2$ values of c aside from $\pm\frac{1}{2}$ for which b is real and different from zero will give $(p-5)/3$ distinct values $a \neq 0$ for which $m_1(a) \sim m_1(0)$. For $a \neq 0$: $c = 1$; $c = -\frac{1}{2}$, $b = \pm\sqrt{-3/4\gamma}$ give $m_1(a) \sim m_1(-\gamma/a)$, and $c = -1$; $c = \frac{1}{2}$, $b = \pm\sqrt{-3/4\gamma}$ give $m_1(a) \sim m_1(-a)$. Now suppose that $\bar{a}$ is not one of the $(p-2)/3$ values a (including 0) for which $m_1(a) \sim m_1(0)$. Then the above mentioned $(p-1)/2 - 2$ values of c for which b is real give $(p-5)/3$ distinct values $\bar{a}'$ which with $-\bar{a}$ and $\gamma/-\bar{a}$ give

$$(p-5)/3 + 2 = (p+1)/3$$

values a for which $m_1(\bar{a}') \sim m_1(\bar{a})$. Since

$$(p-2)/3 + (p+1)/3 + (p+1)/3 = p$$

there are for $p = 6k+5$ just three τ-classes given by the solutions $m_1(a)$.

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[21] L. E. Dickson, *Bulletin of the American Mathematical Society*, vol. 13 (1906), 1 ff.

[22] L. E. Dickson, *Linear Groups*, Chapter IV.

Class 23) $f(x)=x_1^2x_3+x_3^2x_2$. The τ-classes (all from $M_6(x_1,0,x_3)$ are given by

$$m_1=\begin{pmatrix}0&0&0\\-1&0&0\\0&0&0\end{pmatrix};\quad m_2=\begin{pmatrix}0&0&0\\-1&0&0\\0&1&0\end{pmatrix};$$

$$m_3=\begin{pmatrix}0&0&0\\-1&0&0\\1&0&0\end{pmatrix};\quad m_4=\begin{pmatrix}0&-1&-1\\-1&0&1\\1&1&0\end{pmatrix}.$$

Class 24) $f(x)=x_1x_2x_3$. The τ-classes are given by:

$$(1)\ \begin{pmatrix}x_1&0&0\\0&x_2&0\\0&0&x_3\end{pmatrix};\quad (2)\ \begin{pmatrix}x_1&0&0\\0&x_2&x_1\\0&0&x_3\end{pmatrix};\quad (3)\ \begin{pmatrix}x_1&0&x_2\\0&x_2&x_1\\0&0&x_3\end{pmatrix};$$

$$(4)\ \begin{pmatrix}x_1&0&0\\0&x_2&x_1\\x_2&0&x_3\end{pmatrix};\quad (5)\ \begin{pmatrix}x_1&x_3&0\\0&ax_2&x_1\\(1-a)x_2&0&x_3\end{pmatrix},\quad \alpha\neq 0,1;\ a \text{ and } (1-a)$$

giving the same τ-class;

$$(6)\ \begin{pmatrix}x_1&x_3&x_2\\0&x_2&x_1\\0&0&x_3\end{pmatrix}.^{23}$$

Class 25) $f(x)=x_1^2x_3-\gamma x_2^2x_3$, $\gamma\neq$ square. The τ-classes are given by:

$$m_1=\begin{pmatrix}0&\gamma&0\\1&0&0\\0&0&0\end{pmatrix};\ \text{and}\ m_2=\begin{pmatrix}0&\gamma&0\\1&0&0\\1&0&0\end{pmatrix}\ \text{from}\ M_5(x_1,0,x_3);$$

and

$$m_3(a)=\begin{pmatrix}-a&0&0\\0&a&a^2-\gamma\\1&0&0\end{pmatrix},\ m_3(a)\overset{\tau}{\sim}m_3(-a),\ \text{from}\ M_6(x_1,0,x_3).$$

Class 26) $f(x)=x_1(x_1+x_3)(x_1-x_3)$. The τ-classes (all from $M_7(x_1,0,x_3)$) are given by

$$m_1=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix};\quad m_2=\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix};$$

$$m_3=\begin{pmatrix}0&1&0\\0&0&0\\1&0&0\end{pmatrix};\quad m_4=\begin{pmatrix}0&\frac12&\frac12\\1&0&1\\1&-1&0\end{pmatrix}.$$

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[23] For the computation in this class we used $f_1(x)=x_1x_3(x_1+x_2)$ and then transformed the results back to $f(x)=x_1x_2x_3$.

Class 27) $f(x) = x_1(x_1^2 - \gamma x_2^2)$, $\gamma \neq$ square. We have two τ-classes:

$$m_1 = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ from } M_1(x_1, 0, x_3), \text{ and } m_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & \gamma \\ 0 & 1 & 0 \end{pmatrix} \text{ from } M_2(x_1, 0, x_3).$$

Class 28) $f(x) = x_1^2(x_1 + x_2)$. The τ-classes are given by:

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad m_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad m_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$m_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad m_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

from $M_1(x_1, 0, x_3)$; and

$$m_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad m_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad m_8 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$m_9 = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad m_{10} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \text{ from } M_2(x_1, 0, x_3).$$

Class 29) $f(x) = x_1^3$. The τ-classes are given by:

$$m_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad m_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}: \quad m_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

from $M_1(x_1, 0, x_3)$; and

$$m_4 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

from $M_2(x_1, 0, x_3)$.

Class 30) $f(x) = x_1^3 - \alpha x_1 x_2^2 + \alpha x_2^3$ an irreducible cubic.[24] There is no τ-class from $M_1(x_1, 0, x_3)$; from $M_2(x_1, 0, x_3)$ we have the single τ-class given by $m_1 = \begin{pmatrix} 0 & 0 & -\alpha \\ -1 & 0 & -\alpha \\ 0 & 1 & 0 \end{pmatrix}$.

[24] Campbell's class 31) is included in class 30).

In treating $f(x) \equiv 0$ we may obtain forms $F(x, y, z)$ which have $l = 3$ but one or both of m and k less than 3. We may in such cases suppose $m \geq k$. No such case will give a group G with $l = 3$, $m = 3$, $k = 3$, but it might give a group G with $l = 6$, $m = 3$, $k = 3$. The groups G for $k < 3$ have been classified.[25] Interpreting these results for the forms here under consideration we get: for 3, 3, 1 one τ-class represented by $\begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which does not give a group G for either $l = 3$ or $l = 6$; for 3, 2, 2 two τ-classes represented by $\begin{pmatrix} x_1 & x_3 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} x_1 & x_2 & 0 \\ x_3 & x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; for 3, 3, 2 we get eight forms which can be derived from those for $M_1, \cdots, M_7$ and M_{10} (§ 7) by interchanging the variables x and z.

For $l = m = k = 3$ we list the following:

$$M^{(0)}(x) = \begin{pmatrix} 0 & x_1 & x_2 \\ x_1 & 0 & x_3 \\ x_2 & -x_3 & 0 \end{pmatrix}, \quad M^{(1)}(x) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix}, \quad M^{(2)}(x) = \begin{pmatrix} 0 & x_1 & x_2 \\ x_1 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix}$$

$M^{(\nu)}(x)$ has ν x-sections of rank one, $\nu = 0, 1, 2$, and the three τ-classes are therefore distinct. We now show that these are the only such τ-classes. If $M(x)$ has no x-section of rank one and $|M(x)| \equiv 0$, $M(x_1, x_2, 0)$ may evidently be taken as M_8, M_9 or M_{10} for $l = 2$. If

$$[M_8] : \left| \begin{pmatrix} x_1 & x_2 & 0 \\ \gamma x_2 & x_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (a_{ij})x_3 \right| = 0$$

we may by the automorphism $x'_1 = a_{11}x_3 + x_1$, $x'_2 = a_{12}x_3 + x_2$ obtain $a'_{11} = a'_{12} = 0$. Since $x_1^2x_3$ cannot appear $a_{33} = 0$. For $x_1x_3^2$ and $x_2x_3^2$ not to appear we must have $a_{23}a_{32} + a_{13}a_{31} = 0$ and $\gamma a_{13}a_{32} + a_{23}a_{31} = 0$ which requires (since $\gamma \neq$ square) either $a_{13} = a_{23} = 0$ or $a_{31} = a_{32} = 0$ both cases being excluded as they give a row or column of zeros. If

$$[M_9] : \left| \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & x_2 \\ 0 & 0 & 0 \end{pmatrix} + (a_{ij})x_3 \right| = 0$$

we may as before suppose $a_{22} = a_{23} = 0$. Then from the terms in $x_1^2x_3$, $x_2^2x_3$, and $x_1x_2x_3$ we get $a_{33} = a_{31} = a_{32} = 0$ and hence a zero row, which excludes the case. If

[25] H. R. Brahana, *American Journal of Mathematics*, vol. 56 (1934), pp. 490-510.

$$[M_{10}] : \left| \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & x_2 \\ x_2 & 0 & 0 \end{pmatrix} + (a_{ij})x_3 \right| = 0,$$

we may suppose $a_{22} = a_{23} = 0$. Then the terms in $x_1{}^2x_3$ and $x_2{}^2x_3$ give $a_{33} = a_{12} = 0$, and the one in $x_1x_2x_3$ gives $a_{13} = -a_{32}$. Then $| a_{ij} | = -a_{13}{}^2a_{21} = 0$ gives $a_{21} = 0$ for otherwise $M(0,0,1)$ is of rank one. Now let $x'_3 = a_{13}x_3$, giving $a'_{13} = 1$. Then the terms in $x_1x_3{}^2$ and $x_2x_3{}^2$ give $a_{11} = a_{31} = 0$. Interchanging the first two rows gives $M^{(0)}(x)$ representing the only τ-class with no x-section of rank one.

A similar consideration of M_{11}, M_{12}, M_{13}, and M_{14} shows that $M^{(1)}$ and $M^{(2)}$ represent the only other τ-classes with $f(x) \equiv 0$ and actually belonging to the case $(3, 3, 3)$.

10. Uniqueness properties of τ-classes corresponding to groups $G(l, 3, 3)$. We have in sections 4-9 determined the τ-classes $(l, 3, 3)$, $l = 0, 1, 2, 3$ which derivation carries with it, as we have seen in section 3, the classifications for $l = 9, 8, 7, 6$. The case $l = 4$ would involve among other things a classification of quaternary cubic forms and will not be treated here.

In section 2, we saw that certain groups G define more than one τ-class. But a group G is completely defined by a single τ-class g. Hence the possibility of defining G by a second τ-class g must be property of the τ-class g itself, entirely aside from its relation to the group. This, indeed, is merely another way of saying that a collection E of forms is defined by any single form in it.

Suppose that the τ-class g defines G with generating subgroups S and U with the required properties (§ 1). In looking for generating subgroups S' and U' of G which might define another τ-class g' we need not consider subgroups obtained from S and U by means of operations under which the τ-class is invariant, viz. isomorphisms of G; new choices of generators in S, U, and K; and τ. The generators of S' and U' may be written in the form $r^xu^ys^z$ where we may, by means of isomorphisms of G, drop the r^x. Suppose now that

$$S' = \{u^{y^{(1)}}s^{z^{(1)}}, \cdots, u^{y^{(k')}}s^{z^{(k')}}\} \quad \text{and} \quad U' = \{u^{\eta^{(1)}}s^{\zeta^{(1)}}, \cdots, u^{\eta^{(m')}}s^{\zeta^{(m')}}\}$$

gives the τ-class g'. The changes induced on the variables and coefficients of the trilinear form will take a form in g into one in g'. The induced change will therefore be a linear transformation on the variables y and z taken together, and of such a nature that the derived form is still linear in the new

sets of variables y' and z'. Then the problem stated in terms of forms alone is: (1) under what circumstances will transformations of the form:

$$y'_i = \sum_{j=1}^{m} a_{ij} y_j + \sum_{j=1}^{k} a_{i,j+m} z_j, \qquad (i = 1, 2, \cdots, m');$$

$$z'_i = \sum_{j=1}^{m} a_{i+m',j} y_j + \sum_{j=1}^{k} a_{i+m',j+m} z_j, \qquad (i = 1, 2, \cdots, k),$$

where (a_{ij}) is non-singular, replace a given trilinear form in variables x, y, z by another trilinear form in variables x, y', z'? (2) under what circumstances when (1) is possible will the derived form belong to a different τ-class from that of the initial form? We shall not attempt to answer these questions in general but will examine the τ-classes $(l, 3, 3)$ and answer question (2) for them. As an aid in this we introduce the concept of *separable* groups (and forms).

DEFINITION. *If generators of a group $G(l, m, k)$ can be chosen so that $r_{ij} = 1$, $(i > m_1, j \leq k_1)$ and $(i \leq m_1, j > k_1)$ we shall say that G is separable into the components*

$$G_1 = \{u_1, \cdots, u_{m_1}, s_1, \cdots, s_{k_1}\} \quad \textit{and} \quad G_2\{u_{m_1+1}, \cdots, u_m, s_{k_1+1}, \cdots, s_k\}.$$

The groups G_1 and G_2 are also groups " G " and may themselves be separable. If so we may continue the separation process until we have in G the *non-separable* or inseparable component groups $G_1, G_2, \cdots, G_\lambda$ no two of which have in common operators outside the commutator subgroup, and such that $G = \{G_1, G_2, \cdots, G_\lambda\}$. Such a separation will be called a *complete separation* of G.

The separable groups $(l, 3, 3)$ are equivalent to the following three:

$$G^{(1)} = \begin{pmatrix} r_{11} & 1 & 1 \\ 1 & r_{22} & 1 \\ 1 & 1 & r_{33} \end{pmatrix}, \qquad G^{(2)} = \begin{pmatrix} r_{11} & 1 & 1 \\ 1 & r_{22} & r_{23} \\ 1 & r_{32} & r_{33} \end{pmatrix}, \qquad G^{(3)} = \begin{pmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & 1 \\ r_{31} & 1 & 1 \end{pmatrix},$$

where the r_{ij} may or may not satisfy further relations.

Given $G = \{s_1, s_2, s_3, u_1, u_2, u_3\}$ and a τ-class g defined by this representation of G we ask if there can exist subgroups Σ and Γ in G satisfying our hypothesis for generating subgroups (§ 1) and such that the τ-class g' defined by the representation $G = \{\Sigma, \Gamma\}$ of G is different from g. This might be possible for some τ-classes and not for others. If Σ is of order $p^{m'}$ and Γ of order $p^{k'}$ we must have $m' + k' = 6$; for $m + k + l$ and l, and therefore

$m + k$ are invariants of G. We may express any generator of Σ or Γ as $r^x u^y s^z$ where, as we have seen, we lose no generality in suppressing the r^x.

First suppose that $\Sigma\{\sigma_1, \sigma_2, \sigma_3\}$ is of order p^3. Then $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ is likewise of order p^3. Let $\sigma_i = u^{y^{(i)}} s^{z^{(i)}}$; $\gamma_i = u^{\eta^{(i)}} s^{\zeta^{(i)}}$; $i = 1, 2, 3$. Now if the $s^{z^{(i)}}$ generate S we may write a) $G = \{\Sigma, U\}$ and the commutator structure will remain unaltered since $u^y s^z$ has the same commutator as s^z with $u^{y'}$. Similarly if: b) $G = \{\Gamma, U\}$, c) $G = \{S, \Sigma\}$ or d) $G = \{S, \Gamma\}$ there is no change in the defining relations. Any change in the τ-class then would have to come in the second step when the other of the new generating subgroups is introduced. Given $G = \{\Sigma, U\}$ we may express the γ_i in terms of the generators of Σ and of U, thus having the problem: given $G = \{S, U\}$ to find Γ so that $G = \{S, \Gamma\}$ etc. But then the $u^{\eta^{(i)}}$ must generate U and the τ-class remains unchanged under the replacement.

The only case then that we need consider is that in which none of a), b), c), d) above are possible, i. e. Σ and Γ each contain at least one operator in common with S and U. By new choice of generators in the four subgroups we may suppose $\Sigma = \{s_1, u^y s^z, u_3\}$, $\Gamma = \{u_1, u^{y'} s^{z'}, s_3\}$. We first require that G be non-separable. Now one of z_2 and z'_2 is different from zero. By at most change of notation, we may suppose $z_2 \neq 0$. Then a new choice of generators gives $s^z = s_2$. Now for Σ and Γ to be abelian we must have $r_{13} = r_{31} = r_{32} = 1$. If $y'_2 \neq 0$ we have also $r_{23} = 1$ and G is separable. Hence, we suppose $y'_2 = 0$. Then $z'_2 y_2 \neq 0$ since Σ and Γ generate G. But this gives $r_{12} = r_{21} = 1$ and G is again separable. Hence, *if G is non-separable it can belong to only one τ-class* $(l, 3, 3)$.

Next suppose Σ of order p^4. Now if $\{s^{z^{(i)}}\} = S$, some operator from U in Σ would be permutable with S contrary to hypothesis on G. Hence, we may choose generators in S, U, and Σ so that $\Sigma = \{s_1, s_2, u_1, u_2\}$; which requires $r_{11} = r_{12} = r_{21} = r_{22} = 1$. Now let $\Gamma = \{u^{y^{(1)}} s^{z^{(1)}}, u^{y^{(2)}} s^{z^{(2)}}\}$. Since $G = \{\Sigma, \Gamma\}$ we cannot have $\begin{vmatrix} y_3{}^{(1)} & y_3{}^{(2)} \\ z_3{}^{(1)} & z_3{}^{(2)} \end{vmatrix} = 0$ so we may suppose $u^{y^{(1)}} = u_3$ and then $y_3{}^{(2)} = 0$, $s_3{}^{(2)} \neq 0$ and by new choice of generators we may suppose $\Gamma = \{u_3 s^{z^{(1)}}, u^{y^{(2)}} s_3\}$ where $z_3{}^{(1)} = y_3{}^{(2)} = 0$. Hence $r_{33} = 1$ and G is separable and isomorphic to one of the groups $G^{(3)}$, listed above. Σ of order p^5 is evidently impossible for G separable or inseparable.

Now consider G separable. The groups $G^{(3)}$ as we have just seen can be defined by τ-classes $(l, 2, 4)$ and hence cannot be isomorphic to the groups $G^{(1)}$ and $G^{(2)}$ unless they can likewise be thus defined. But in neither $G^{(1)}$ nor $G^{(2)}$ are there pairs of permutable subgroups of order p^2 one each from S and U. Further the groups $G^{(3)}$ are the only ones which would be found among

the groups $G(l, m, k)$ for two different solutions of $m + k = 6$. These groups were classified by Brahana.[26] Since there is at most one τ-class for each value of l, the groups $G^{(3)}$ present no uniqueness problem.

The groups $G^{(2)}$ and $G^{(1)}$ are readily distinguished by various properties (characteristic subgroup structure etc.) and so none of them can be defined by two different τ-classes. Our conclusion then is that except for the three groups $G^{(3)}$ for $l = 2, 3, 4$ each group $G(l, 3, 3)$ defines just one τ-class.

11. Deep lying nature of group theoretic properties derived from trilinear form invariants. For an interpretation of some of the form invariants in terms of the groups suppose that $r_1, \cdots, r_l$ generate K and that $u_i^{-1}s_j^{-1}u_is_j = \prod_h r_h^{a_{hij}}$. Now we ask conditions on z so that s^z will be permutable with some operator in U. For this it is necessary and sufficient that the commutators of s^z with $u_1, \cdots, u_m$ generate a group of order less than p^m. u_i and s^z have the commutator $\prod_h r_h^{\sum_j a_{hij}z_j}$. Hence, for these commutators to be related we must have z such that the matrix $M_z = (\sum_j a_{hij}z_j)$ with m rows and l columns be of rank less than m. For $l < m$ this is true for all z. For $l = m = k = 3$ we have as our condition $| M_z | = 0$, i. e. z is a point on the z-cubic related to the form. It will be permutable with u^y where y must be on the y-cubic. If M_z is of rank two s^z will be permutable with just one subgroup u^y. If M_z is of rank one s^z will be permutable with a subgroup of order p^2 in U. If no z-section is of rank less than three evidently no y-section could be of rank less than three, for if so some u^y would be permutable with s^z and hence M_z would be of rank less than three. But M_z is symmetric in x and y. Hence, if $| M_z | = 0$ is a cubic without real points, then $f(x) = | M_x | = 0$ is a cubic without real points and conversely. So for $m = k = l = 3$ we have as a necessary and sufficient condition that no operator in S be permutable with any operator in U that $f(x)$ be the "imaginary" cubic. (Class 20).

Proceeding similarly we could obtain numerous theorems of this character. However, from the complexity of the considerations involved it seems unlikely that we will be able to express in ordinary group theoretic terms the non-isomorphism of many of the groups which we have proved non-isomorphic by the above algebraic considerations. For instance, consider the two groups defined by the matrix

$$M_x(x) = \begin{pmatrix} x_1 & x_3 & 0 \\ 0 & x_1 & x_3 \\ 0 & 0 & x_1 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & -1-a & 0 \end{pmatrix}$$

[26] H. R. Brahana, *American Journal of Mathematics*, vol. 57 (1935), pp. 645-667.

for two values a_1 and a_2 of a, $a_i \neq 0, -1$; $a_1 + a_2 \neq -1$. We have proved that these groups G_1 and G_2 are non-isomorphic. Let us review the properties which these groups have in common. They are metabelian groups of the same order, p^9; both conformal with the abelian group of order p^9 and type $1, 1, 1, \cdots$; each contains maximal invariant subgroups of order p^6 which with an abelian subgroup of its group of isomorphisms will generate the whole group. Generators of G_1 and G_2 can be so chosen that the totality of operators s^z permutable with operators in U is the same in each case and also the totality of operators u^y permutable with operators in S is the same in each case. In both groups every commutator different from identity is of the first kind and extent two. Furthermore the subgroups are the same in the sense that if we consider all of the subgroups of G_1 isomorphic to a given one there will be the same number of subgroups in G_2 isomorphic to the named one. The quotient groups are similarly related. The difference between the two groups lies in the different arrangements relative to each other of the subgroups.

This situation is somewhat analogous to the projective classification of sets of five (no three collinear) points in the plane. Two such sets may be projectively non-equivalent and yet the subsets of the two are projectively equivalent.

A complete classification of the groups G then from a group theoretic standpoint must involve the introduction of new group theoretic conceptions, perhaps an extension of the "type" and "extent" defined above.

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BIHARMONIC FUNCTIONS IN ABSTRACT SPACES.*

By A. E. TAYLOR.

1. Introduction. In classical analysis there is a remarkable parallelism between the theory of analytic functions of a complex variable and the theory of harmonic functions of two real variables; the source of this relationship lies in the Cauchy-Riemann equations. When we consider analytic functions of n complex variables there is a corresponding parallelism with a class of functions of $2n$ real variables. This class is usually designated by the name biharmonic; it is much more restricted than the class of solutions of Laplace's equation in $2n$ dimensions. In fact, a biharmonic function of $x_1, \cdots, x_n, y_1, \cdots, y_n$ satisfies the system of n^2 equations.[1]

$$\frac{\partial^2 u}{\partial x_\mu \partial x_\nu} + \frac{\partial^2 u}{\partial y_\mu \partial y_\nu} = 0 \qquad \frac{\partial^2 u}{\partial x_\mu \partial y_\nu} - \frac{\partial^2 u}{\partial x_\nu \partial y_\mu} = 0. \tag{1}$$

It has been shown elsewhere that the Cauchy-Riemann equations may be generalized to meet the needs of the theory of analytic functions on one normed vector space to another.[2] It is the purpose of this note to make the natural extension of the notion of a biharmonic function which is suggested by the generalization just mentioned, and to prove a fundamental theorem pertaining to such functions. As an illustration we consider real-valued biharmonic functions of the doubly infinite set $(x_1, x_2, \cdots; y_1, y_2, \cdots)$ where each of the sequences $x = \{x_n\}, y = \{y_n\}$ is taken to be a point of the Hilbert space (l_2), that is, $\sum_1^\infty |x_n|^2$ is finite, and similarly for $\{y_n\}$.

2. The Cauchy-Riemann equations. Let E, E' be real, Banach spaces. With E we can associate a complex space $E(C)$ of couples of elements from E. If (x, y) is such a couple, and a, b are real numbers, we define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(a + ib) \cdot (x, y) = (ax - by, bx + ay)$$
$$\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}.$$

* Received September 28, 1937.

[1] W. F. Osgood, *Lehrbuch der Funktionentheorie*, vol. II, 1 (1929), pp. 22-23.

[2] A. E. Taylor, "Analytic functions in general analysis," *Annali della Reale Scuola Normale Superiore di Pisa*, (in press) § 8. We shall refer to this as paper (A). See also *Comptes Rendus*, vol. 203 (1936), pp. 1228-1230.

Two couples are regarded as equal if and only if their corresponding members are equal. If we write x for $(x, 0)$ we may also write $(x, y) = x + iy$, as with complex numbers. We shall denote $x + iy$ by the single letter z. Clearly $E(C)$ is also a Banach space (complex). In a similar fashion we construct $E'(C)$.

We may also form a *real* space of these same couples, defining the product $a(x, y)$ only when a is real: $a(x, y) = (ax, ay)$, and norming it in the same way. This space, which we denote by E^2, is likewise complete; as a metric space it is indistinguishable from $E(C)$.

Let $f(z) = f_1(x, y) + if_2(x, y)$ be a function defined on an open set D of $E(C)$, its values being in $E'(C)$. *It is said to be analytic in D if it is continuous there and possesses a variation at each point of D.*[3] If $f(z)$ is analytic the essential properties of the functions f_1, f_2 may be stated as follows (Theorem 17, paper (A)).

THEOREM 1. *In order that $f(z)$ be analytic in D it is necessary and sufficient that f_1, f_2 be continuous and admit continuous first partial variations satisfying the equations*

$$\begin{aligned} \delta_\xi{}^x f_1(x, y) &= \delta_\xi{}^y f_2(x, y) \\ \delta_\xi{}^y f_1(x, y) &= -\delta_\xi{}^x f_2(x, y) \end{aligned} \tag{2}$$

at all points of D, for an arbitrary element ξ of E. Both f_1 and f_2 then admit continuous total Fréchet differentials of all orders in D.[4]

3. Biharmonic functions. Let us first recall an important symmetry property of Fréchet differentials: if $F(x)$ is a function on one complete normed vector space to another which is defined in the neighborhood of x_0, and possesses continuous first and second Fréchet differentials there, then $d_{y_2}{}^x d_{y_1}{}^x F(x) = d_{y_1}{}^x d_{y_2}{}^x F(x)$ at each point of the neighborhood.[5] If we apply this to the functions f_1, f_2 of § 2 we find symmetry relations of the type

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[3] A function $F(x)$ is said to have a variation at x_0 if it is defined in the neighborhood of x_0 and if $\delta_y{}^x F(x_0) = \lim\limits_{t \to 0} \dfrac{F(x_0 + ty)}{t}$ exists for every y. The variable t is real or complex according as the spaces are real or complex.

[4] By the *total* differential of $f_1(x, y)$ is meant the differential with respect to the composite variable (x, y), an element of the space E^2. The partial Fréchet differentials of $f_1(x, y)$ exist also, and coincide with the partial variations, so that the total differential is

$$d_{(\xi, \eta)}{}^{(x,y)} f_1(x, y) = \delta_\xi{}^x f_1(x, y) + \delta_\eta{}^y f_1(x, y).$$

See T. H. Hildebrandt and L. M. Graves, *Transactions of the American Mathematical Society*, vol. 29 (1927), pp. 136-138. We use the Latin d for Fréchet differentials.

[5] M. Kerner, *Annals of Mathematics*, vol. 34 (1933), p. 549.

$$d_\eta{}^x d_\xi{}^x f_1(x,y) = d_\xi{}^x d_\eta{}^x f_1(x,y), \qquad d_\eta{}^x d_\xi{}^y f_1(x,y) = d_\xi{}^y d_\eta{}^x f_1(x,y)$$

and certain others which need not be written down because of their similarity. Making use of these, we find upon differentiating equations (2) that the functions f_1, f_2 must satisfy the conditions

$$\begin{aligned} d_\eta{}^x d_\xi{}^x f_k(x,y) + d_\eta{}^y d_\xi{}^y f_k(x,y) &= 0 \\ d_\eta{}^x d_\xi{}^y f_k(x,y) - d_\eta{}^y d_\xi{}^x f_k(x,y) &= 0 \end{aligned} \qquad (k = 1, 2) \tag{3}$$

at each point of D, for each element (ξ, η) of E^2. These equations are evidently a generalization of (1), and are equivalent to the latter if E and E' coincide with the real Euclidean n-space. Accordingly we lay down the following definition:

Definition. A function $u(x,y)$ which is defined in an open set D of E^2, with values in E', is said to be biharmonic in D if it is continuous in D and possesses first and second total Fréchet differentials which are continuous in D and there satisfy equations (3).

In order to be assured of the appropriateness of this terminology it is necessary to show that a biharmonic function is derivable from a suitable analytic function on $E(C)$ to $E'(C)$, of which it is the 'real' part. We shall do this. First we must consider an existence theorem for what may be called 'exact differentials.'

THEOREM 2. *Let D be a simply connected*[6] *open set in the space E^2. Let $P(x,y,\xi)$, $Q(x,y,\xi)$ be functions with values in E', defined when (x,y) is in D and ξ is in E; furthermore let P and Q be linear in ξ, and possess continuous Fréchet differentials with respect to (x,y) such that for (x,y) in D, ξ_1, ξ_2 in E*

$$\begin{aligned} d_{\xi_2}{}^x P(x,y,\xi_1) &= d_{\xi_1}{}^x P(x,y,\xi_2) \\ d_{\xi_2}{}^y Q(x,y,\xi_1) &= d_{\xi_1}{}^y Q(x,y,\xi_2) \\ d_{\xi_2}{}^y P(x,y,\xi_1) &= d_{\xi_1}{}^x Q(x,y,\xi_2). \end{aligned} \tag{4}$$

Then there exists a function $F(x,y)$ on D to E', unique apart from an additive constant, whose total Fréchet differential exists and is precisely $P(x,y,\xi) + Q(x,y,\eta)$.

This theorem is an immediate consequence of a theorem of Kerner.[7] By means of it we now readily establish our principal result:

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[6] We use the definition given by Kerner, *loc. cit.*, p. 555.

[7] M. Kerner, *loc. cit.*, p. 555, Theorem 3.

THEOREM 3. *Let $u(x,y)$ be biharmonic in a simply connected open set D of the space E^2. Then there exists a second function $v(x,y)$ which is also biharmonic in D, and such that the couple $(u,v) = u + iv$, regarded as a function on $E(C)$ to $E'(C)$, is analytic in D. The function $v(x,y)$ is unique apart from an additive constant.*

Proof. Consider the functions

$$P(x,y,\xi) = -d_\xi{}^y u(x,y),\, Q(x,y,\eta) = d_\eta{}^x u(x,y).$$

They satisfy the hypotheses of Theorem 2 by virtue of (3) and the properties of Fréchet differentials. Therefore there is defined in D a function $v(x,y)$ with the continuous differential $-d_\xi{}^y u(x,y) + d_\eta{}^x u(x,y)$. But also, the differential of $v(x,y)$ is $d_\xi{}^x v(x,y) + d_\eta{}^y v(x,y)$. Hence u and v are continuous, and together satisfy the Cauchy-Riemann equations (2); therefore $u + iv$ is analytic in D. From this it follows that v is also biharmonic.

4. Biharmonic functions of an infinite number of variables. Let E be the Hilbert space (l_2) and E' the real number system. Then $E(C)$ is the complex Hilbert space H_0 analagous to (l_2), and E^2 is the space of couples (x,y), where $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots)$. The typical element of H_0 is $z = x + iy = (x_1 + iy_1, x_2 + iy_2, \cdots)$. Let D be an open set in H_0 (we may also regard it as an open set in $(l_2)^2$). If $f(z) = f_1(x,y) + if_2(x,y)$ is a complex function defined on D the conditions for analyticity may be stated as follows.

THEOREM 4. *In order that $f(z)$ be analytic in D it is necessary and sufficient that $f(z)$ be continuous in D and possess first partial derivatives $\frac{\partial f}{\partial z_\nu}$ $(\nu = 1, 2, \cdots)$ at each point of D. Stated in terms of f_1 and f_2 the conditions are: f_1 and f_2 shall be continuous in D and possess continuous first partial derivatives with respect to each of the real variables x_ν, y_ν,* and *in addition, the equations*

$$\text{(5)} \qquad \frac{\partial f_1}{\partial x_\nu} = \frac{\partial f_2}{\partial y_\nu} \qquad \frac{\partial f_1}{\partial y_\nu} = -\frac{\partial f_2}{\partial x_\nu} \qquad (\nu = 1, 2, \cdots)$$

must be satisfied.

Proof. We consider the conditions on $f(z)$ first. They are clearly necessary. They are also sufficient. It is enough to prove this for an open set D of the type $\| z \| < r$. Corresponding to each z of this set and each positive integer n we consider the function $f_n(z) = f(z^{(n)})$, where

$$z^{(n)} = (z_1, z_2, \cdots, z_n, 0, \cdots).$$

Since $\| z^{(n)} \| \leq \| z \|$ these functions are continuous in D, and hence analytic, for it is easily seen that they possess the variations

$$\delta_w{}^z f_n(z) = \sum_{\nu=1}^{n} f_{z_\nu}(z^{(n)}) w_\nu \qquad \left(f_{z_\nu} = \frac{\partial f}{\partial z_\nu} \right).$$

The analyticity of $f(z)$ in D is then a consequence of the fact that as n tends to infinity, $f_n(z) \to f(z)$, the convergence being uniform in each compact set extracted from the closed sphere $\| z \| \leq \theta r$ $(0 < \theta < 1)$. For let G be such a compact set, and let $\epsilon > 0$ be given. Denote by G' the set consisting of all points of G together with all points $z^{(n)}$ $(n = 1, 2, \cdots)$ where z is in G. Then it is not difficult to see that G' is compact and lies in the same closed sphere with G. Now $f(z)$ is uniformly continuous in the compact set G' (paper (A), Theorem 4). Let us choose δ so that $\| z' - z \| < \delta$ implies $| f(z') - f(z) | < \epsilon$ whenever z, z' are in G'. Fréchet has shown [8] that corresponding to the compact set G' there exists a convergent series $\sum_1^\infty a_\nu{}^2$ such that $\sum_{\nu=n+1}^{\infty} | z_\nu |^2 < \sum_{\nu=n+1}^{\infty} a_\nu{}^2$ for all n and all z in G'. Hence if we choose N so that $n \geq N$ implies $\sum_{\nu=n+1}^{\infty} a_\nu{}^2 < \delta^2$ we shall have $\| z^{(n)} - z \|^2 = \sum_{\nu=n+1}^{\infty} | z_\nu |^2 < \delta^2$, and $| f(z) - f_n(z) | < \epsilon$ for all z in G whenever $n \geq N$. $f(z)$ is then analytic, by an extension of a theorem of Weierstrass (Theorem 13 of paper (A)). Since it is known that the variation of an analytic function is analytic and is, in fact, its Fréchet differential, we are able to infer that the partial derivatives of $f(z)$ are analytic, and that $d_w{}^z f(z) = \sum_{\nu=1}^{\infty} \frac{\partial f}{\partial z_\nu} w_\nu$. The remainder of the theorem is now deducible by classical methods.

Finally we consider conditions analogous to (1) which will assure us that a real function $u(x, y)$ defined on an open set D in $(l_2)^2$ is biharmonic there.

THEOREM 5. *Let $u(x, y)$ be defined and continuous in D, and possess continuous first and second partial derivatives with respect to the variables x_ν, y_ν. Let*

$$a_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad b_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial y_\nu}, \quad c_{\mu\nu} = \frac{\partial^2 u}{\partial y_\mu \partial y_\nu}$$

and suppose that:

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[8] M. Fréchet, *Rendiconti dell Circolo Matematico di Palermo*, vol. 30 (1910), pp. 18-19.

(i) *the series* $\sum_{\nu=1}^{\infty}\left[\left(\frac{\partial u}{\partial x_\nu}\right)^2+\left(\frac{\partial u}{\partial y_\nu}\right)^2\right]$ *converges uniformly in each compact set contained in an arbitrary closed sphere in* D;[9]

(ii) *each of the series* $\sum_{\nu=1}^{\infty} a^2_{\mu\nu}$, $\sum_{\nu=1}^{\infty} b^2_{\mu\nu}$, $\sum_{\nu=1}^{\infty} c^2_{\mu\nu}$ $(\mu=1,2,\cdots)$ *converges according to condition* (*c*) *in* D;

(iii) *for each* ξ *in* (l_2) *each of the sequences*

$$\sum_{\mu=1}^{n}\Big|\sum_{\nu=1}^{\infty} a_{\mu\nu}\xi_\nu\Big|^2, \qquad \sum_{\mu=1}^{n}\Big|\sum_{\nu=1}^{\infty} b_{\mu\nu}\xi_\nu\Big|^2, \qquad \sum_{\mu=1}^{n}\Big|\sum_{\nu=1}^{\infty} c_{\mu\nu}\xi_\nu\Big|^2$$

converges according to condition (*c*) *in* D;
(iv) *the equations*

$$a_{\mu\nu}+c_{\mu\nu}=0 \qquad b_{\mu\nu}-b_{\nu\mu}=0$$

are satisfied at each point of D (*compare with* (1) *of* § 1). *Then:* u *has continuous first and second total Fréchet differentials satisfying* (3) *of* § 3 *in* D, *and hence is biharmonic.*

Proof. We can at once infer, by methods due to Hart,[10] that the first differential exists, is continuous in D, and given by

$$d_{(\xi,\eta)}{}^{(x,y)}u(x,y)=\sum_{\nu=1}^{\infty}\left(\frac{\partial u}{\partial x_\nu}\xi_\nu+\frac{\partial u}{\partial y_\nu}\eta_\nu\right). \tag{6}$$

This all follows as a consequence of the hypotheses on u and its first partial derivatives. The next thing is to prove that the function

$$\phi(x,y,\xi,\eta)=d_{(\xi,\eta)}{}^{(x,y)}u(x,y),$$

when ξ,η are fixed, has these same properties. The existence and continuity of the partial derivatives of ϕ in D follows by classical methods, because of (ii) and the symmetry of the matrices $\{a_{\mu\nu}\}$, $\{b_{\mu\nu}\}$ and $\{c_{\mu\nu}\}$; $\{b_{\mu\nu}\}$ is symmetric, by (iv). We have

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[9] We shall then, for brevity, say that the series converges according to condition (c) in D.

[10] W. L. Hart, *Transactions of the American Mathematical Society*, vol. 23 (1922), pp. 30-39. Our assumptions, though somewhat weaker than Hart's are equally effective, for by them we are enabled to show that the series in condition (i) defines a function which is bounded in each of the compact sets in question, and continuous in D; similarly for the function in (6) when ξ,η are fixed.

$$\frac{\partial\phi}{\partial x_\mu} = \sum_{\nu=1}^{\infty} a_{\mu\nu}\xi_\nu + \sum_{\nu=1}^{\infty} b_{\mu\nu}\eta_\nu$$
$$\frac{\partial\phi}{\partial y_\mu} = \sum_{\nu=1}^{\infty} b_{\nu\mu}\xi_\nu + \sum_{\nu=1}^{\infty} c_{\mu\nu}\eta_\nu ,$$

from which it follows rather easily by (iii) that the series $\sum_{\mu=1}^{\infty}\left[\left(\frac{\partial\phi}{\partial x_\nu}\right)^2 + \left(\frac{\partial\phi}{\partial y_\mu}\right)^2\right]$ converges according to condition (c) in D, for each ξ. From (iii) it also follows, by a theorem of Hellinger and Toeplitz,[11] that the matrices $\{a_{\mu\nu}\}$, $\{b_{\mu\nu}\}$, $\{c_{\mu\nu}\}$ define bilinear forms which are, at each point of D, bounded in the sense of Hilbert. By repetition of the reasoning already referred to in obtaining the first differential of $u(x, y)$ we see that

$$d_{(\xi',\eta')}{}^{(x,y)}\phi(x, y, \xi, \eta) = \sum_{\mu,\nu=1}^{\infty} a_{\mu\nu}\xi'_\mu\xi_\nu + \sum_{\mu,\nu=1}^{\infty} b_{\mu\nu}\xi'_\mu\eta_\nu + \sum_{\mu,\nu=1}^{\infty} b_{\nu\mu}\eta'_\mu\xi_\nu + \sum_{\mu,\nu=1}^{\infty} c_{\mu\nu}\eta'_\mu\eta_\nu .$$

The remainder of the theorem then follows at once by use of condition (iv) and the definition of biharmonicity.

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[11] *Mathematische Annalen*, vol. 69 (1910), pp. 321-322. See also M. H. Stone and J. D. Tamarkin, *Duke Mathematical Journal*, vol. 3 (1937), p. 298.

NORMAL COORDINATES FOR EXTREMALS TRANSVERSAL TO A MANIFOLD.*

By STEWART S. CAIRNS.

1. Introduction. Consider a positive definite regular calculus of variations problem defined on an n-dimensional manifold R throughout a neighborhood of a point q_0. A normal coördinate system [1] with origin q_0 is a system (y) in terms of which the extremals with q_0 for initial point can be represented, near the initial point, by linear equations $y^i = \lambda_i s$ $(0 \leq s < s_0)$.

In the present paper, we first obtain normal coördinates under hypotheses weaker than those heretofore used. We then define a new kind of coördinates, $(z) \equiv (z^1, \cdots, z^n)$, called *normal coördinates with respect to M near q_0*, where M is an m-dimensional manifold on R passing through q_0. In terms of (z), M is defined near q_0 by the equations $z^{m+1} = \cdots = z^n = 0$, and the general extremal cut transversally by M at its initial point, near q_0, is given by $z^i = a_i$ $(i = 1, \cdots, m)$, $z^j = \lambda_j s$ $(j = m + 1, \cdots, n)$ $(0 \leq s < s_0)$ where s represents arc length and where $\sum_{j=m+1}^{n} \lambda_j \lambda_j = 1$.

Underlying much of our work is a study by Morse [2] of the extremals cut transversally at their initial points by a manifold M. The subset of these extremals consisting of all those with a given initial point q covers an $(n - m)$-manifold which is differentiable near q save, in general, for a conical point at q. We obtain necessary and sufficient conditions on our calculus of variations problem that this manifold be differentiable even at q, independently of the particular manifold M.

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[1] An existence proof for normal coördinates is partly given in Duschek-Mayer, *Lehrbuch der Differentialgeometrie*, vol. 2, ch. V, §§ 5, 6, and is completed by J. H. C. Whitehead, "On the covering of a complete space by the geodesics through a point (§ 2)," *Annals of Mathematics*, vol. 36 (1935), pp. 679-704. The original relevant investigations both of the extremals with given initial point and of transversal extremals were made, in the case where R is euclidean 3-space, by Bliss and Mason, "The properties of curves in space which minimize a definite integral," *Transactions of the American Mathematical Society*, vol. 9 (1908), pp. 440-466; "Fields of extremals in space," *Transactions of the American Mathematical Society*, vol. 11 (1910), pp. 325-340.

[2] Marston Morse, "The calculus of variations in the large," *American Mathematical Society Colloquium Publications*, vol. 18 (1934), p. 111.

2. Definition of the metric. By an *n-manifold* ($n = 1, 2, \cdots$), we mean a connected topological space which can be covered by a denumerable set of neighborhoods, each the homeomorph of an open region of euclidean n-space. Let coördinate systems $(x), (y), \cdots$ be introduced, by homeomorphisms, on the neighborhoods of such a set. The manifold is said to be *of class* C^ν in terms of the coördinate systems $(x), (y), \cdots$, which are called *admissible systems*, if every transformation between two of the systems is given by functions $x^i(y)$ of class [3] C^ν with a non-vanishing jacobian. Any further coördinate system is *admitted* if the transformations between it and the original systems are all of class C^ν with non-vanishing jacobians.

We suppose that R is an n-manifold ($n > 1$) of class C^2. Let q_0 be a point on R and let (x_0) denote the coördinates of q_0 in some admissible system (x). Consider a calculus of variations problem whose basic function,[4] $F(x, r)$, is defined for (x) in some neighborhood of (x_0) and for $(r) \neq 0$. We suppose that F and F_{r^i} are of class C^1, that F is positive homogeneous of the first order in (r), and that

$$F(x, r) > 0, \qquad F_1(x, r) \neq 0 \qquad \text{when } (r) \neq 0, \tag{2.1}$$

where F_1 is defined (Morse, *op. cit.*, p. 112) by the identity

$$\begin{vmatrix} F_{r^i r^j} & u_i \\ v_j & 0 \end{vmatrix} \equiv -F_1(x, r)[r^i u_i][r^j v_j]. \tag{2.2}$$

Our variations problem defines a metric, $ds = F(x, dx)$, on any region of R throughout which the above conditions are fulfilled.

3. Normal coördinates with origin q_0. By a *unit* contravariant vector (ρ) at (x), we mean one satisfying the equation

$$F(x, \rho) = 1. \tag{3.1}$$

There is a unique extremal tangent at a given initial point (x_0) to a given vector (r_0). We fix the parameter, t, on this extremal by requiring that $F(x, \dot{x}) = F(x_0, r_0)$ and that $x^i(0) = x_0^i$. The extremal, with its parameter, is then determined by the following system of equations, in which (ρ_0) denotes the unit vector in the same direction as (r_0):

$$\begin{aligned} &\frac{d}{dt} F_{r^i}(x, \dot{x}) - F_{x^i}(x, \dot{x}) = 0, \\ &x^i(0) = x_0^i, \qquad \dot{x}^i(0) = r_0^i = \tau\rho_0^i, \\ &F(x, \dot{x}) = F(x_0, \tau\rho_0) = \tau. \end{aligned} \tag{3.2}$$

---

[3] That is, continuous along with all derivatives of orders $\leq \nu$.

[4] For the laws of transformation of this function, see Morse, *loc. cit.*

The parameter t is related to the arc length, s, measured from (x_0) by the identity

$$s = \tau t = F(x_0, r_0)t. \tag{3.3}$$

Following a method to be found in Morse [*op. cit.*, Ch. V, § 4], we next consider the system

$$\frac{d}{dt}[(1+\lambda)F_{r^i}(x, \dot{x})] - F_{x^i}(x, \dot{x}) = 0, \quad F(x, \dot{x}) = \tau, \quad (\tau > 0), \tag{3.4}$$

where τ is a constant. If, in a solution $[x^i(t), \lambda(t)]$, we have $\lambda(0) = 0$, then $\lambda(t)$ is identically zero. [For proof, see Morse, *loc. cit.*].

When $F = \tau$, we have

$$\begin{vmatrix} F_{r^i r^j} & F_{r^i} \\ F_{r^j} & 0 \end{vmatrix} = -\tau^2 F_1 \neq 0. \tag{3.5}$$

It is therefore possible, by the implicit function theorem, to solve the equations

$$(1+\lambda)F_{r^i}(x, r) = v^i, \qquad F(x, r) = \tau \tag{3.6}$$

for $(r^1, \cdots, r^n, \lambda)$ in a neighborhood of $\lambda = 0$. The solutions,

$$r^i = r^i(x, v, \tau) = \tau r^i(x, v, 1), \qquad \lambda = \lambda(x, v, \tau) \tag{3.7}$$

are of class C^1.

In place of the system (3.2), we shall use the system

$$\begin{aligned} &\frac{dx^i}{dt} = r^i(x, v, \sigma) = \sigma r^i(x, v, 1), \\ &\frac{dv_i}{dt} = q_i(x, v, \sigma) = F_{x^i}(x, r(x, v, \sigma)) = \sigma q_i(x, v, 1), \\ &x^i(0) = x_0{}^i, \\ &v_i(0) = v_{i0} = F_{r^i}(x_0, r_0), \end{aligned} \tag{3.8}$$

where σ is a positive constant. We write the general solution for x^i in the system (3.8) as follows:

$$x^i = h^i(t, \sigma, x_0, v_0). \tag{3.9}$$

In the case where $\sigma = \tau = F(x_0, r_0)$, the system (3.8) is equivalent to (3.4) with the initial conditions $[x^i(0) = x_0{}^i, \lambda(0) = 0]$ and is therefore equivalent to (3.2), since $\lambda(0) = 0$ implies that $\lambda(t)$ is identically zero. Hence, the extremals with q_0 for initial point are given by

$$\begin{aligned} x^i &= h^i(t, \tau, x_0, v_0) \\ &= h^i[t, F(x_0, r_0), x_0, F_{r^i}(x_0, r_0)] \equiv \phi^i(t, x_0, r_0). \end{aligned} \tag{3.10}$$

In view of identity (3. 3), these equations can be written in the form

$$
(3.11)\qquad
\begin{aligned}
x^i &= \phi^i\left(\frac{s}{\tau}, x_0, \rho_0\tau\right)\\
&= \phi^i\left(\frac{s}{F(x_0, r_0)}, x_0, r_0\right) \equiv \psi^i(s, x_0, r_0).
\end{aligned}
$$

The functions ψ^i are solutions for (x) in the system (3. 8) read for the special case where (r_0) is a unit vector and $\sigma = 1$.

(A) *The above method reveals the class C^1 character of (ϕ^i, ψ^i) and $(\phi_t{}^i, \psi_s{}^i)$ near (x_0), since the functions r^i in* (3. 7) *and* (3. 8) *are of class C^1. We note the following properties:*

$$
(3.12)\qquad
\begin{aligned}
&\phi^i(0, x_0, r_0) = \psi^i(0, x_0, r_0) = x_0{}^i,\\
&\phi_t{}^i(0, x_0, r_0) = r_0{}^i,\\
&\psi_s{}^i(0, x_0, r_0) = \rho_0{}^i.
\end{aligned}
$$

Let $s_0 > 0$ be so small that the representation (3. 11) is valid $0 \leq s < s_0$ for all (ρ_0). The representation (3. 10) is then valid $0 \leq t < s_0/\tau$. With (x_0, ρ_0) held fast, τ and r_0 can be so restricted in (3. 10) and (3. 11) that $\tau = F(x_0, r_0)$ satisfies the condition $0 \leq \tau < s_0$. This will result in no loss of generality in the results we have in mind. Under this restriction, the point where $t = 1$ is on the domain of (3. 10). Since it coincides with the point where $s = \tau$, we have the identity

$$
(3.13)\qquad \psi^i(s, x_0, r_0) \equiv \phi^i(1, x_0, \rho_0 s) \qquad (0 < s < s_0),
$$

which holds on each extremal with (x_0) for initial point. Hence, if we make the definition $\phi^i(1, x_0, 0) = x_0{}^i$, we can write the equations of all the extremals with q_0, for initial point in the form

$$
(3.14)\qquad x^i = \phi^i(1, x_0, \rho_0 s) \equiv g^i(x_0, \rho_0 s) \qquad (0 \leq s < s_0).
$$

We now [cf. Mayer, *loc. cit.*] define the *normal coördinates* (y) with origin q_0 by the transformation

$$
(3.15)\qquad x^i = g^i(x_0, y).
$$

This transformation is continuous throughout some neighborhood of (x_0). With the possible exception of the point (x_0), the partial derivatives exist and are continuous throughout such a neighborhood. The existence of these derivatives at (x_0) with the values

$$\left.\frac{\partial g^i}{\partial y^j}\right|_{(y)=(0)} = \delta_j{}^i \tag{3.16}$$

is established as follows. Let $\rho_{(j)}$ denote either of the unit vectors at q_0 with all (x)-components equal to zero save the j-th. Then at (x_0)

$$\begin{aligned} \lim_{\Delta y^j \to 0} \frac{\Delta_{y^j} g^i}{\Delta y^j} &= \lim_{s\to 0} \frac{g^i(x_0, s\rho_{(j)}) - g^i(x_0, 0)}{s\rho^j{}_{(j)}} \\ &= \frac{1}{\rho^j{}_{(j)}} \lim_{s\to 0} \frac{dg^i}{ds} = \frac{\rho^i{}_{(j)}}{\rho^j{}_{(j)}} = \delta_j{}^i. \end{aligned} \tag{3.17}$$

To establish the continuity of $\partial g^i/\partial y^j$ at $(y) = (0)$, we first note that

$$\frac{dg^i}{ds} = \rho_0{}^j \left(\frac{\partial g^i}{\partial y^j}\right), \tag{3.18}$$

since equation (3.15) is obtained from (3.13) by the substitution $(y) = (\rho_0 s)$. But dg^i/ds is the same as $\psi_s{}^i(s, x_0, r_0)$. Hence

$$\rho_0{}^j \left(\frac{\partial g^i}{\partial y^j}\right) = \psi_s{}^i(s, x_0, r_0). \tag{3.19}$$

The continuity of $\partial g^i/\partial y^j$ at $(y) = (0)$ follows readily from (A) together with equations (3.16) and (3.19). Furthermore, the jacobian $| \partial g^i/\partial y^j |$ has the value unity when $(y) = (0)$.

(B) *The transformation* (3.15) *to normal coördinates is therefore of class C^1 with a non-vanishing jacobian in some neighborhood of $(y) = (0)$. In terms of (y), the extremals with q_0 for initial point are given, near q_0, by $y^i = \rho^i s$ $(0 \leq s < s_0)$, where s is arc length and (ρ) ranges over all the unit vectors at q_0.*

4. Transversal vectors. Let M be an m-manifold $(0 < m < n)$ of class C^2 on R. This means that M is intrinsically of class C^2, and also that, if $(x) = (x^1, \cdots, x^n)$ and $(\alpha) = (\alpha^1, \cdots, \alpha^m)$ are admissible coördinate systems on R and M respectively, having some point q_0 common to their domains, then, in a neighborhood of q_0 on M, we have

$$x^i = \phi^i(\alpha) \qquad (i = 1, \cdots, n), \tag{4.1}$$

where the ϕ's are of class C^2 and the functional matrix $| \partial\phi^i/\partial\alpha^j | \equiv | \phi_j{}^i(\alpha) |$ is of rank m.

If (r^*) and (r) denote contravariant vectors at (x), then we say that (r^*) is *transversal* to (r) if the invariant equation

$$F_{r^i}(x, r) r^{*i} = 0 \tag{4.2}$$

holds. If (x) is on M and every vector (r^*) tangent to M at (x) is transversal to (r), we say that M is *transversal* to (r) at (x). The condition that M be transversal to (r) at (x) can be expressed in terms of the ϕ's as follows:

$$F_{r^i}[\phi(\alpha), r]\phi_h{}^i(\alpha) = 0 \qquad (h = 1, \cdots, m). \tag{4.3}$$

The solutions (r) of (4.3) will be called the *transversal vectors at* (α).

In the space of the vectors (r), the equation

$$F(x, r) = 1 \tag{4.4}$$

represents a closed convex [5] $(n-1)$-dimensional manifold of class C^2, called the *indicatrix* of our calculus of variations problem in the point (x). The following statement gives a geometric interpretation of transversality.

(A) Let (ρ) be a unit vector at $(x) \equiv [\phi(\alpha)]$, and let S_{n-1} be the indicatrix in the point (x). Let (τ_m) denote the m-plane tangent to M at (α). Then (ρ) is a unit transversal vector at (α) if and only if the $(n-1)$-plane tangent to S_{n-1} at (ρ), where (ρ) is now interpreted as a point on S_{n-1}, contains an m-plane parallel to τ_m.

(B) Let 0 be the origin in the space of the vectors (r) at (x) and let $S_{n-m-1}(\tau_m)$ be the set of all points on S_{n-1} at each of which the tangent $(n-1)$-plane to S_{n-1} contains an m-plane parallel to τ_m. Then the rays from 0 through $S_{n-m-1}(\tau_m)$ form a cone, referred to hereafter as the *transversal cone*, whose elements give the directions of the transversal vectors at (α).

5. The polar coördinates (α, λ, s).

We next establish the following result.

THEOREM. *The transversal cone is the homeomorph of an $(n-m)$-plane.*

Proof. We first define *unit covariant vectors* at q near q_0 with reference to a set $\pi_i{}^{(j)}(q)$ $(j = 1, \cdots, n)$ of n covariant vectors, arbitrarily selected subject to the restrictions (1) that they be independent at each point q in some neighborhood of q_0 and (2) that their components be class C^1 functions of the (x)-coördinates of q. A further restriction will be imposed later. By a *unit covariant vector* π_i at q, we will mean any vector of the form

$$\pi_i = \lambda_j \pi_i{}^{(j)}(q) \qquad (i = 1, \cdots, n), \tag{5.1'}$$

where

$$\lambda_j \lambda_j = 1. \tag{5.1''}$$

[5] Cf. C. Carathéodory, *Variationsrechnung* (1935), §§ 289-293.

The symbols (ρ^i, π_i) will be used for unit contravariant and covariant vectors, respectively, and (r^i, p_i) will be used for general contravariant and covariant vectors.

LEMMA. *The equations*

$$F_{r^i}(x, \rho) - \kappa\pi_i = 0, \quad F(x, \rho) = 1, \quad (\kappa > 0) \qquad (i = 1, \cdots, n), \tag{5.2}$$

in which (π) *is a unit covariant vector at* (x), *can be solved for* κ *and the* ρ*'s, thus:*

$$\begin{aligned} \kappa &= \kappa\,(x^1, \cdots, x^n; \pi_1, \cdots, \pi_n) \\ \rho^i &= \rho^i(x^1, \cdots, x^n; \pi_1, \cdots, \pi_n) \end{aligned} \tag{5.3}$$

where the functions κ *and* ρ^i *are of class* C^1 *and the correspondence* (5.3) *is one-to-one between the totality of unit contravariant and covariant vectors* (ρ) *and* (π) *at* (x).

For a proof of essentially this lemma, see Morse, *op. cit.*, p. 241.

(A) *Let functions* $r^i(x, p)$ *be defined by the identities*

$$r^i(x, \tau\pi) = \tau\rho^i(x, \pi) \qquad (\tau > 0), \tag{5.4}$$

(π) *being a unit covariant vector. Then*

$$r^i = r^i(x, p) \tag{5.5}$$

is a one-to-one correspondence between all the covariant and all the contravariant vectors, (p) *and* (r) *respectively, at* (x). *This correspondence agrees with* (5.3) *when* (p) *is a unit vector.*

Now consider the correspondence (5.5) for points $(x) = [\phi(\alpha)]$ on M and for vectors (p) which satisfy the linear homogeneous equations

$$p_i\phi_h{}^i(\alpha) = 0 \qquad (h = 1, \cdots, m). \tag{5.6}$$

(B) We restrict the vectors $\pi^{(j)}(q)$ used in the definition of unit covariant vectors [cf. (5.1)] by the requirement that, when q is at (α) on M, the first $(n-m)$ of these vectors constitute a set $\pi^{(j)}(\alpha)$ $(j = 1, \cdots, n-m)$ of independent solutions of (5.6), so selected as to be of class C^1 in (α).

The unit covariant vector solutions of (5.6) for fixed (α) are

$$\pi_i(\alpha) = \lambda_j\pi_i{}^{(j)}(\alpha), \qquad \lambda_j\lambda_j = 1, \qquad (j = 1, \cdots, n-m). \tag{5.7}$$

This set of vectors is topologically equivalent to an $(n-m-1)$-sphere. The

general solution of (5.6) is $p_i(\alpha) = \tau\lambda_j\pi_i{}^{(j)}(\alpha)$ as τ ranges over all positive numbers. From the above lemma, together with equations (4.3), (5.6), and (5.7), we see that the transversal contravariant vectors (r) at (α) are the vectors

(5.8) $$r^i = r^i[\phi(\alpha), \tau\lambda_j\pi^{(j)}(\alpha)], \qquad \lambda_j\lambda_j = 1 \qquad (\tau > 0),$$

and our theorem follows at once.

We will say that an extremal [cf. equations (3.12)]

(5.9) $$x^i = g^i(x_0, \rho_0 s)$$

is *cut transversally by M at its initial point*, or *issues transversally from M*, if (x_0) is on M and (ρ_0) is a unit transversal vector at (α). The set of all extremals issuing transversally from M is given, near q_0, by the equations

(5.10) $$\begin{aligned} x^i &= g^i[\phi(\alpha), sr(\phi, \lambda_j\pi^{(j)}(\alpha))] \\ &= g^i[\phi, r(\phi, s\lambda_j\pi^{(j)})], \end{aligned}$$

where

(5.11) $$\lambda_j\lambda_j = 1, \qquad 0 \leq s < s_0$$

s_0 being a sufficiently small positive quantity.

(C) *The quantities* $(\alpha, \lambda, s) \equiv (\alpha^1, \cdots, \alpha^m, \lambda_1, \cdots, \lambda_{n-m}, s)$ *are a coördinate system near* q_0. *For a fixed set of values* (α), *the quantities* (λ, s) *constitute a polar coördinate system on the* $(n-m)$*-dimensional cone whose elements are the extremals issuing transversally from M at* (α). *A set of values* (α, λ) *specifies one of these extremals, on which s is the arc length. In terms of* (α, λ, s), *the equation of M near q_0 is $s = 0$. The transformation* (5.10) *between* (x) *and* (α, λ, s) *is of class C^1 save when $s = 0$.*

6. Flat transversal cones. A point q on M is generally a conical point of the $(n-m)$-manifold covered by the extremals issuing transversally from M at q. In certain important special cases, which we now investigate, this $(n-m)$-manifold has no conical point at q but has there a unique tangent $(n-m)$-plane. This is equivalent to saying that the transversal vectors at q lie in an $(n-m)$-plane.

THEOREM. *Given m, the transversal vectors to every differentiable m-manifold at any point where the conditions of § 2 are fulfilled will always lie in an $(n-m)$-plane if and only if the following condition is satisfied:* (1) *in case $m = n-1$, that*

(6.1) $$F(x, r) \equiv F(x, -r),$$

(2) *in case* $0 < m < n - 1$, *that* F *be of the form*

$$F(x, r) = \sqrt{a_{ij}(x) r^i r^j}. \tag{6.2}$$

With the aid of the geometric interpretation of transversality [§ 4, (A) and (B)], we see that this theorem is a consequence of the following lemmas applied to the indicatrix, S_{n-1}. We prove the first lemma in a somewhat more general form than necessary, because of its geometric interest.

LEMMA 1. *Let* S_{n-1} *be a differentiable* $(n - m)$*-manifold in an affine* n*-space. Suppose every ray from a certain point,* O*, meets* S_{n-1} *in a single point. Suppose further that, for a given positive integer* $m < n$ *and for every* m*-plane* τ_m *through* O*, the point set* $S_{n-m-1}(\tau_m)$ [For this notation, see § 4 (B)] *is the intersection of* S_{n-1} *with an* $(n - m)$*-plane* $E_{n-m}(\tau_m)$ *through* O*. Then* S_{n-1} *is symmetric in* O *and, if* $m < n - 1$*, is a hyperellipsoid.*

LEMMA 2. *In the same* n*-space, if* S_{n-1} *is a closed convex differentiable* $(n-1)$*-manifold, symmetric in* O*, then* $S_0(\tau_{n-1})$ *is, for every* τ_{n-1}*, the intersection of* S_{n-1} *with a line through* O*. If* S_{n-1} *is a hyperellipsoid, center at* O*, then* $S_{n-m-1}(\tau_m)$ *is always* $(m = 1, \cdots, n-1)$*, the intersection of* S_{n-1} *with an* $(n - m)$*-plane through* O*.*

Proof. We prove only Lemma 1, since Lemma 2 presents no difficulty. We give the proof with the aid of six subsidiary results, of which (I)-(IV) are in the present section.

(I) If the hypothesis of Lemma I holds for a given value $m < n - 1$, then it holds with $(m + 1)$ in place of m.

For, let τ_{m+1} be any $(m + 1)$-plane, and let (τ_m^0, τ_m^1) be two non-parallel m-planes on τ_{m+1}. Then $S_{n-m-2}(\tau_{m+1})$ is the common part of $S_{n-m-1}(\tau_m^0)$ and $S_{n-m-1}(\tau_m^1)$ and is therefore the intersection of S_{n-1} with the linear space E in which $E_{n-m}(\tau_m^0)$ and $E_{n-m}(\tau_m^1)$ intersect. It is easy to verify that $S_{n-m-2}(\tau_{m+1})$, as the set of points where the tangent $(n-1)$-plane to S_{n-1} contains a parallel to τ_{m+1}, is of dimensionality at least $(n - m - 2)$. Hence E is of dimensionality at least $(n - m - 1)$. We must also show that E is of dimensionality at most $(n - m - 1)$, in other words that $E_{n-m}(\tau_m^0)$ and $E_{n-m}(\tau_m^1)$ do not coincide. This follows from the fact that, if E_{n-1} is an $(n-1)$-plane containing τ_m^0 and not containing τ_m^1, the points where the tangent plane to S_{n-1} is parallel to E_{n-1} belong to $S_{n-m-1}(\tau_m^0)$ and not to $S_{n-m-1}(\tau_m^0)$.

(II) As a consequence of result (I), it is sufficient to prove Lemma 1 in the two cases $m = n - 1, n - 2$.

The following statements are easy to verify.

(III) Under the hypothesis of Lemma 1, for any $(n-m)$-plane, E_{n-m}, through O, there exists an m-plane, τ_m, through O such that

$$E_{n-m}=E_{n-m}(\tau_m). \tag{6.3}$$

(IV) Let S^*_1 be the curve in which S_{n-1} is met by an arbitrary given 2-plane, E_2, through O. Lemma 1 will follow in all its generality if we show that S^*_1 is symmetric in O and, in case $m=n-2$, is an ellipse.

We first note that no $(n-1)$-plane, τ_{n-1}, through O can be tangent to S_{n-1}. For, by the hypothesis of Lemma 1 in the case $m=n-1$, $S_0(\tau_{n-1})$ consists of exactly two points. The tangent planes at these points can be obtained by moving an $(n-1)$-plane, keeping it parallel to τ_{n-1}, in either direction from τ_{n-1} to the last positions in which it meets S_{n-1}. Hence neither point of $S_0(\tau_{n-1})$ lies on τ_{n-1}. It follows that no ray from O can be tangent to S_{n-1}. For if Q were a point of tangency of such a ray, then the $(n-1)$-plane tangent to S_{n-1} at Q would contain the ray and hence pass through O. We can accordingly represent the curve S^*_1 in terms of polar coördinates (ρ,ϕ) on E_2 with the pole at 0, by an equation

$$\rho=K(\phi), \tag{6.4}$$

where $K(\phi)$ is differentiable. Furthermore,

$$K(\phi+2\pi)=K(\phi). \tag{6.5}$$

From the hypotheses of the lemma it follows that the tangents to S^*_1 at the opposite ends of any chord through O are parallel. This condition can be expressed in the form

$$\frac{d}{d\phi}\log K(\phi)=\frac{d}{d\phi}\log K(\phi+\pi). \tag{6.6}$$

From equations (6.5) and (6.6), it follows that $K(\phi)$ is periodic of period π, hence that S^*_1 is symmetric in O.

Lemma 1 *is now established in the case* $m=n-1$.

7. The case of an ellipsoidal indicatrix.[6] The present section completes the work of § 6 by establishing § 6, Lemma 1, in the case $m=n-2$ [cf. § 6 (II)]. In this case, (6.3) becomes

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[6] The work of this section is largely a generalization of a proof in the case $(m,n)=(1,3)$ by W. Blaschke, "Räumliche Variationsprobleme mit symmetrischen Transversalitätsbedingungen," *Leipzig. Berichte*, vol. 68 (1916), pp. 50-55.

$$E_2 = E_2(\tau_{n-2}) \tag{7.1}$$

and, by our definitions,

$$S^*_1 = S_1(\tau_{n-2}). \tag{7.2}$$

Let τ^0_{n-1} be an $(n-1)$-plane containing τ_{n-2}. By § 6 (I) and the lemma for the case $m = n-1$, the points where the tangent plane to S_{n-1} is parallel to τ^0_{n-1} are the end points (P, Q) of a chord of S_{n-1} through O.

(V) Let τ_{n-1} be the $(n-1)$-plane parallel to τ^0_{n-1} through a variable point, O', of the chord PQ. Then the intersection, S_{n-2}, of S_{n-1} with τ_{n-1} satisfies, in the space τ_{n-1}, the hypothesis of § 6, Lemma 1, with $(n-2, n-1)$ in place of (m, n) and with O' in place of O.

Proof. We first regard τ_{n-2} as free to assume any position through O on the fixed $(n-1)$-plane τ^0_{n-1}. Then $E_2(\tau_{n-2})$ always passes through (P, Q). Conversely, if $E_2 = E_2(\tau_{n-2})$ is any 2-plane through (P, Q), then $\tau_{n-2} \subset \tau^0_{n-1}$. As a consequence of our definitions, the intersection of $S_1(\tau_{n-2})$ with S_{n-2} is the set of points where the tangent $(n-2)$-plane to S_{n-2} is parallel to τ_{n-2}. These are also the points in which S_{n-2} is met by the line common to τ_{n-1} and $E_2(\tau_{n-2})$. For the complete establishment of result (V), it remains only to show that this line, as τ_{n-1} varies, always meets $S_1(\tau_{n-2})$ in just two points (A, B). When τ_{n-1} is at τ^0_{n-1}, this follows from the definition of S_{n-1}. As τ_{n-1} moves from τ^0_{n-1}, towards P for example, (A, B) may be thought of as varying continuously. If there were a first position in which the line AB met $S_1(\tau_{n-2})$ in a third point, C, we should then have AB tangent to $S_1(\tau_{n-2})$ at C. But C would then be a point where the tangent to S_{n-1} is parallel to τ_{n-1} and, by § 6 (I) and Lemma 1 for $n - m = 1$, (P, Q) are the only such points. The proof of result (V) is now complete.

(VI) In the case $m = n-2$, $S^*_1 = S_1(\tau_{n-2})$ is an ellipse.

Proof. From the proof of (V), together with § 6, Lemma 1, in the case $n - m = 1$, we see that each chord of the set AB of parallel chords of $S_1(\tau_{n-2})$ is bisected by PQ. We refer to PQ as a *line of symmetry* and to the common direction of the chords AB as the *corresponding direction.* Now, keeping τ_{n-2} fixed, let τ^0_{n-1} adopt every position such that $\tau^0_{n-1} \supset \tau_{n-2}$. Then PQ varies correspondingly and adopts the position of every line in E_2 through O. Accordingly, every such line is a line of symmetry of $S_1(\tau_{n-2})$. Let (u, v) be a coördinate system in E_2, origin at O, such that the u-axis is in the corresponding direction to the v-axis, regarded as a line of symmetry. Let (L, L') be a second pair of lines through O, where L' is in the corresponding direction to L and where (L, L') are of positive and negative slopes respectively with reference

to the (u, v)-system. By a compression in the v-direction, we can transfer to a new coördinate system (u, v'), in terms of which L and L' have negative reciprocal slopes. Let ζ be the inclination of L, figured as if (u, v') were a rectangular cartesian coördinate system, and suppose L so chosen that ζ is an irrational multiple of π. Let (ρ, ϕ) be the polar coördinate system superposed in the usual way on (u, v'). Then, using $\rho = K(\phi)$ to represent $S_1(\tau_{n-2})$, we have, by symmetry in the lines $\phi = \pi/2$ and $\phi = \zeta$,

$$K(\phi) = K(\pi - \phi), \qquad K(\phi) = K(2\zeta - \phi). \tag{7.3}$$

Hence

$$K(\phi) = K(\pi - 2\zeta + \phi). \tag{7.4}$$

But $K(\phi)$ is also periodic of period π, and the periods π and $\pi - 2\zeta$ are not rational multiples of one another. Hence K is a constant and $S_1(\tau_{n-2})$ is a circle about O in terms of the (u, v')-system. Therefore, in our affine n-space, $S^*_1 = S_1(\tau_{n-2})$ is an ellipse. By § 6 (IV), this completes the proof.

8. Normal coördinates with respect to *M*. (A) *If* $F(x, r) \equiv F(x, -r)$, *then, in the notation of* (5.5),

$$r^i(x, p) \equiv r^i(x, -p). \tag{8.1}$$

If $F(x, r) = \sqrt{a_{ij}(x) r^i r^j}$, *then* $r^i(x, p)$ *is linear homogeneous in* $(\bar\kappa p_1, \cdots, \bar\kappa p_n)$, *where* $\bar\kappa$ *is the function* κ *of equations* (5.3) *figured for the unit covariant vector at* (x) *in the direction of* (p).

Result (A) can be directly verified in equations (5.2)–(5.4).

Our *normal coördinates* (z) *with respect to* M *near* q_0 are defined by the transformation

$$x^i = G^i(z), \tag{8.2}$$

where $G^i(z)$ is obtained by substituting $(z^1, \cdots, z^m)$ for $(\alpha^1, \cdots, \alpha^m)$ and $(z^{m+1}, \cdots, z^n)$ for $(s\lambda_1, \cdots, s\lambda_{n-m})$, respectively, in the functions g^i of equations (5.10).

(B) *In general, the functions* $G^i(z)$ *are of class*[7] C^1 *save when* $z^{m+1} = \cdots = z^n = 0$. *If the condition of* § 6, *Theorem, is fulfilled, these functions are of class* C^1 *without exception and have a non-zero jacobian throughout some neighborhood of* q_0.

The first part of (B) follows from the work of § 5 and the second part follows from (A) above.

[7] Or of higher class, if we strengthen the differentiability assumptions on (M, R, F).

In terms of the normal coördinates (z), M is defined near q_0 by the equations

$$z^{m+1} = \cdots = z^n = 0, \tag{8.3}$$

and the extremals issuing transversally from a point $(z) = (a^1, \cdots, a^m, 0, \cdots, 0)$ of M near q_0 are defined, near M, by the equations

$$\begin{aligned} &z^i = a^i & & (i = 1, \cdots, m), \\ &z^{m+j} = \lambda_j s \qquad (0 \leq s < s_0) & & (j = 1, \cdots, n - m), \end{aligned} \tag{8.4}$$

where s represents arc length and where

$$\lambda_j \lambda_j = 1. \tag{8.5}$$

If the condition of § 6, Theorem, is satisfied, we have the reversible case, and all the extremals which are cut transversally by M, not necessarily at their initial points, are given near q_0 by equations (8.4), with s permitted to take on negative as well as positive values. If the condition of § 6, Theorem, is not satisfied, then in general an extremal cut transversally by M at a point q will be of class C^2 in terms of (z) save for a corner at q, and one of the two parts into which q divides it will be a ray in (z)-space.

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PROBLEMS OF CLOSEST APPROXIMATION ON A TWO-DIMENSIONAL REGION.*

By Dunham Jackson.

1. Introduction. The theorems of Markoff and Bernstein on the derivatives of polynomials and trigonometric sums can be made to serve as basis for a theory of the convergence of certain types of polynomial and trigonometric approximation to functions of a single real variable.[1] It is fairly apparent that similar methods can be applied to functions of two variables. In the carrying out of this process there are so many possible variations and combinations that it would not be profitable to enumerate the resulting theorems systematically, still less to discuss them successively in detail. Nevertheless, the extension involves some adjustments which are not entirely automatic or superficial, and it has been thought worth while to present below some typical results illustrating the differences between the one-dimensional and two-dimensional formulations.[2]

There is occasion first to develop two-dimensional versions of the Markoff and Bernstein theorems themselves.

2. Theorems of Markoff and Bernstein for polynomials. Let $P(x, y)$ be a polynomial of the n-th degree[3] in the variables x and y together. (It is then of the n-th degree in each variable separately; on the other hand, an arbitrary polynomial of degree n in each variable is of degree $2n$ in both together, and the discussion as given is applicable on replacement of n by $2n$.)

If $|P(x, y)| \leqq L$ at all points of a straight line segment of length h in the (x, y)-plane, and if $\partial P/\partial s$ denotes directional differentiation along the line,

$$\left| \frac{\partial P}{\partial s} \right| \leqq \frac{2n^2 L}{h}$$

* Received November 23, 1937.

[1] See for example D. Jackson, "Certain problems of closest approximation," *Bulletin of the American Mathematical Society*, vol. 39 (1933), pp. 889-906; "Bernstein's theorem and trigonometric approximation," *Transactions of the American Mathematical Society*, vol. 40 (1936), pp. 225-251.

[2] See also E. Carlson, "On the convergence of trigonometric approximations for a function of two variables," *Bulletin of the American Mathematical Society*, vol. 32 (1926), pp. 639-641; E. L. Mickelson, "On the approximate representation of a function of two variables," *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 759-781.

[3] This expression will be understood throughout to mean "of the n-th degree at most."

on the entire segment, and

$$\left| \frac{\partial P}{\partial s} \right| \leqq \frac{nL}{[\delta(h-\delta)]^{\frac{1}{2}}}$$

at a point of the segment whose distance from the nearer end is δ. For a transformation of coördinates by translation and rotation to a new system with one axis along the specified line reduces $P(x, y)$ on the line to a polynomial of the n-th degree in a single variable, having L as an upper bound for its absolute value on the segment, and having $\partial P/\partial s$ for its derivative with respect to the variable, so that the upper bounds for $|\partial P/\partial s|$ are given by the standard forms of the Markoff and Bernstein theorems respectively.

If $|P(x, y)| \leqq L$ throughout the square $-1 \leqq x \leqq 1$, $-1 \leqq y \leqq 1$, application of Markoff's theorem to the polynomials in one variable obtained by holding the other variable fast gives $|\partial P/\partial x| \leqq n^2 L$, $|\partial P/\partial y| \leqq n^2 L$ on the square. If $\partial P/\partial s$ is the directional derivative at any point of the square in a direction making an arbitrary angle α with the x-axis,

$$\left| \frac{\partial P}{\partial s} \right| = \left| \frac{\partial P}{\partial x} \cos\alpha + \frac{\partial P}{\partial y} \sin\alpha \right| \leqq n^2 L(|\cos\alpha| + |\sin\alpha|) \leqq 2^{\frac{1}{2}} n^2 L.$$

(Here and in the next statement it is sufficient, as regards the degree of $P(x, y)$, that it be of the n-th degree in each variable separately.) At a point of the square whose shortest distance from the boundary is δ, Bernstein's theorem similarly gives [4]

$$\left| \frac{\partial P}{\partial x} \right| \leqq \frac{nL}{[\delta(2-\delta)]^{\frac{1}{2}}}, \quad \left| \frac{\partial P}{\partial y} \right| \leqq \frac{nL}{[\delta(2-\delta)]^{\frac{1}{2}}}, \quad \left| \frac{\partial P}{\partial s} \right| \leqq \frac{2^{\frac{1}{2}} nL}{[\delta(2-\delta)]^{\frac{1}{2}}}.$$

It is perhaps not necessary at the present stage to formulate the corresponding statements for a rectangle of arbitrary size, shape, and orientation.

Suppose that a point D lies on two mutually perpendicular line segments, on each of which $|P(x, y)| \leqq L$, the length of each segment being $\geqq h$. If ξ, η are coördinates with respect to a pair of axes along the two lines,

$$|\partial P/\partial \xi| \leqq 2n^2 L/h, \qquad |\partial P/\partial \eta| \leqq 2n^2 L/h$$

at the point D, and if $\partial P/\partial s$ is any directional derivative at the point,

$$|\partial P/\partial s| \leqq 2^{3/2} n^2 L/h.$$

[4] In case the horizontal and vertical distances from the boundary are different it is to be noted that if $0 < \delta < \delta' \leqq 1$,

$$\delta(2-\delta) < \delta'(2-\delta'),$$

the expression $\delta(2-\delta)$ as a function of δ having its maximum for $\delta = 1$.

More generally, suppose that the segments are oblique to each other, and let γ denote the magnitude of the acute angle formed by their lines (even if the segments themselves terminate at D and form an obtuse angle there). Let ξ, η be coördinates referred to a pair of rectangular axes with origin at D, the ξ-axis extending along one of the given lines, let P_γ denote the result of differentiation along the other given line, and let P_s be an arbitrary directional derivative at D, in a direction making an angle α with the ξ-axis. Then, with an appropriate choice of the sense of differentiation on each line (the discussion being concerned essentially only with the *magnitudes* of the derivatives),

$$| P_\xi | \leqq 2n^2L/h, \qquad | P_\gamma | \leqq 2n^2L/h,$$
$$P_\gamma = P_\xi \cos\gamma + P_\eta \sin\gamma, \qquad P_\eta = [-P_\xi \cos\gamma + P_\gamma]/\sin\gamma,$$
$$P_s = P_\xi \cos\alpha + P_\eta \sin\alpha = [P_\xi \sin(\gamma - \alpha) + P_\gamma \sin\alpha]/\sin\gamma,$$
$$| P_s | \leqq 4n^2L/(h \sin\gamma).$$

Let R be a closed region (more generally, any point set) for which there are two positive constants h, γ such that each point of R lies on two line segments belonging entirely to R, each of length $\geqq h$, their lines making with each other a minimum angle $\geqq \gamma$. If $| P(x, y) | \leqq L$ throughout R,

$$| \partial P/\partial s | \leqq 4n^2L/(h \sin\gamma)$$

at all points of R (including the boundary),[5] for all directions of differentiation.

If $| P(x, y) | \leqq L$ on two segments which cross at D, the length of each being $\geqq h$, and the angle between them γ, as before, and if D is at a distance $\geqq \delta$ from each end of each segment,

$$\left| \frac{\partial P}{\partial s} \right| \leqq \frac{2nL}{[\delta(h-\delta)]^{\frac{1}{2}} \sin\gamma} \tag{1}$$

for differentiation in any direction at D. If $| P(x, y) | \leqq L$ throughout a region R of the sort described in the last paragraph, and if $0 < \delta \leqq \frac{1}{2}h$, (1) holds for an arbitrary directional derivative at any point of R whose minimum distance from the boundary is $\geqq \delta$. Somewhat less explicitly, $| \partial P/\partial s | \leqq CnL/\delta^{\frac{1}{2}}$, where C depends only on h and γ. If R_1 is a closed point set interior to R, there is a constant C' for R and R_1 such that $| \partial P/\partial s | \leqq C'nL$ throughout R_1.

3. Theorems of Markoff and Bernstein for trigonometric sums. In the preceding discussion repeated use has been made of the fact that a trans-

[5] Both hypothesis and conclusion with regard to $P(x, y)$ are such that if they hold at interior points of R they necessarily hold on the boundary.

formation of coördinates carries a polynomial of the n-th degree into a polynomial of the n-th degree. This property is not shared by trigonometric sums in two variables, and for application to such functions the argument has to be reconsidered accordingly.

Some conclusions, to be sure, which involve no rotation of axes, can be obtained immediately from the corresponding theorems in one variable. Let $T(x, y)$ be a trigonometric sum in x and y which is of the n-th order (i. e., of the n-th order at most) in each variable separately. If $|T(x, y)| \leqq L$ for all values of x and y, $|\partial T/\partial x| \leqq nL$, $|\partial T/\partial y| \leqq nL$, and $|\partial T/\partial s| \leqq 2^{\frac{1}{2}}nL$ for differentiation in any direction at any point. If $|T(x, y)| \leqq L$ throughout a rectangle with sides parallel to the coördinate axes, $|\partial T/\partial s| \leqq Cn^2L$ throughout the rectangle, the constant C depending only on the dimensions of the rectangle,[6] and $|\partial T/\partial s|$ has a constant multiple of nL as upper bound in any closed region interior to the rectangle.[7]

Suppose now that $T(x, y)$ is of the n-th order in the two variables together, and that $|T(x, y)| \leqq L$ on a line segment of length h *with rational slope* p/q, the numbers p and q being integers and relatively prime. Let (x_0, y_0) be a point of the segment, let $(p^2 + q^2)^{\frac{1}{2}} = r$, and let

$$\xi = (q/r^2)(x - x_0) + (p/r^2)(y - y_0),$$
$$\eta = (p/r^2)(x - x_0) - (q/r^2)(y - y_0).$$

The given segment lies in the line $\eta = 0$. The inverse transformation is

$$x = x_0 + q\xi + p\eta,$$
$$y = y_0 + p\xi - q\eta.$$

A trigonometric sum of order n in x and y is for $\eta = 0$ a trigonometric sum of order np_0 in ξ, if p_0 is the larger of p and q; a term $\cos n_1x \cos n_2y$, for example, where $n_1 + n_2 = n$, becomes

$$\cos n_1(x_0 + q\xi) \cos n_2(y_0 + p\xi)$$

of order $n_1q + n_2p \leqq np_0$. If Δs denotes distance in the (x, y) plane, $\Delta s = (\Delta x^2 + \Delta y^2)^{\frac{1}{2}} = r\Delta\xi$ on the line $\eta = 0$, and the difference between the values of ξ corresponding to the ends of the given segment is h/r. Consequently[8]

$$\left|\frac{\partial T}{\partial \xi}\right| \leqq G_0n^2L, \qquad \left|\frac{\partial T}{\partial s}\right| = \left|\frac{1}{r}\frac{\partial T}{\partial \xi}\right| \leqq Gn^2L,$$

[6] See for example D. Jackson, *Transactions, loc. cit.*, p. 230, Theorem 2.

[7] *Ibid.*, p. 227, Theorem 1.

[8] *Transactions, loc. cit.*, Theorem 2.

when G_0 and G depend only on h and r (since $p_0 \leqq r$). At a point of the segment whose distance from the nearer end as measured in the (x, y) plane is not less [9] than δ,

$$\left| \frac{\partial T}{\partial s} \right| \leqq \frac{G_1 n L}{[\delta(h - \delta)]^{\frac{1}{2}}},$$

the constant G_1 also depending only on h and r. As $h - \delta \geqq \frac{1}{2}h$ it may be omitted from the denominator, with a corresponding change in the value of G_1.

The derivative in an arbitrary direction at a point common to two segments making an angle with each other, on each of which $| T(x, y) | \leqq L$, can then be dealt with as in the polynomial case.

Let R' be a region (or more general point set) for which there are three constants $h > 0, \gamma' > 0, r_0$ such that each point of R' lies on two line segments belonging to R', each of length $\geqq h$ and with a rational slope p/q satisfying the condition that $p^2 + q^2 \leqq r_0^2$, while the lines of the segments make with each other a minimum angle $\geqq \gamma'$. If $| T(x, y) | \leqq L$ throughout R', $| \partial T/\partial s | \leqq Cn^2 L$ for differentiation in an arbitrary direction at any point of R', the constant C depending only on h, r_0, and γ'; since the hypothesis admits only a finite number of pairs of values of p and q, and so a finite number of values of $(p^2 + q^2)^{\frac{1}{2}} = r$, the various constants G of the second paragraph preceding which correspond to a fixed h and different values of r can be replaced by a single one equal to the largest of them. At a point whose minimum distance from the boundary is $\geqq \delta$, $0 < \delta \leqq \frac{1}{2}h$,

$$| \partial T/\partial s | \leqq C_1 n L/\delta^{\frac{1}{2}},$$

with a value of C_1 depending only on h, r_0, and γ'. As in the second paragraph preceding the manner of dependence of the constants G and G_1 on h was left wholly unspecified, it is to be noted that under the present hypotheses any segment entering into the discussion whose length is greater than h can be replaced by a segment of length h exactly. If R_1 is a closed point set interior to R', there is a constant C' determinate for R' and R_1 such that $| \partial T/\partial s | \leqq C' n L$ throughout R_1.

The conditions imposed on R in the preceding section and those on R' here will be satisfied by *any region R for which there are two positive constants h, γ such that every point of R is vertex of a circular sector of radius $\geqq h$ and angle $\geqq \gamma$, belonging entirely to R.* A value of r_0 meeting the requirements in the case of R' can be associated with any value of $\gamma' < \gamma$.

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[9] *Ibid.*, Theorem 1.

Some of the results obtained above will be used in the following section, while others have been inserted here merely for purposes of comparison.

4. Approximation by trigonometric sums and polynomials. Trigonometric approximation over the entire plane for a function periodic in both variables has been treated from the point of view of the present article by another writer[10] The ordinary form of Bernstein's theorem in a single variable, applied with respect to x and y separately, can be used to justify the following assertion:

If $f(x, y)$ is a continuous function which is of period 2π in each variable, if $T_n(x, y)$ is a trigonometric sum of the n-th order in each variable (not necessarily of the n-th order in both together), if

$$G_{ns} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} | f(x, y) - T_n(x, y) |^s \, dxdy,$$

s being any positive exponent, and if there exists a trigonometric sum $t_n(x, y)$, of the n-th order in each variable, such that

$$| f(x, y) - t_n(x, y) | \leqq \epsilon_n$$

everywhere, then

$$| f(x, y) - T_n(x, y) | \leqq 4(4n^2 G_{ns})^{1/s} + 5\epsilon_n$$

for all values of x and y.

The proof is closely parallel to that of a corresponding proposition in one variable,[11] the most notable difference, giving rise to the factor $4n^2$ inside the parentheses in the last inequality, being that an interval $| x - x_0 | \leqq 1/(2n)$, of length $1/n$, which enters into the proof in the case of one variable, is to be replaced here by a square $| x - x_0 | \leqq 1/(4n)$, $| y - y_0 | \leqq 1/(4n)$, of area $1/(4n^2)$. (More generally, if the trigonometric sums are of order m in one variable and of order n in the other, n^2 is to be replaced by mn.) If $T_n(x, y)$ is chosen among all trigonometric sums of the order indicated so as to minimize the integral G_{ns}, the fact that G_{ns} does not exceed the corresponding integral with $T_n(x, y)$ replaced by $t_n(x, y)$ implies that $G_{ns} \leqq 4\pi^2\epsilon_n^s$, and $| f(x, y) - T_n(x, y) |$ consequently does not exceed a constant multiple of $n^{2/s}\epsilon_n$. This in conjunction with theorems establishing the existence of trigonometric sums $t_n(x, y)$ giving a specified degree of approximation[12] leads to conclusions as to the uniform convergence of the minimizing sum $T_n(x, y)$

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[10] See E. Carlson, *loc. cit.*

[11] See D. Jackson, *Bulletin*, *loc. cit.*, pp. 899-900, Lemma 5.

[12] See E. L. Mickelson, *loc. cit.*, § 4.

toward $f(x, y)$. The conclusions can be generalized by the introduction of a weight function and by the use of Hölder's inequality in connection with the integral to be minimized.[13] Similar remarks apply to other propositions which are to be formulated below.

If the property of periodicity is dropped, and the hypotheses are made to refer to a rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, the trigonometric sums $T_n(x, y)$ and $t_n(x, y)$ being replaced by polynomials $P_n(x, y)$ and $p_n(x, y)$, of the n-th degree in each variable, with

$$G_{ns} = \int_c^d \int_a^b | f(x,y) - P_n(x,y) |^s \, dxdy,$$

it is found that

$$| f(x,y) - P_n(x,y) | \leqq 4 \left[\frac{64 n^4 G_{ns}}{(b-a)(d-c)} \right]^{1/s} + 5\epsilon_n$$

throughout the rectangle. The factor n^4 in the right-hand member of the inequality, in place of n^2, results from the use of Markoff's theorem instead of that of Bernstein.

Let the domain of integration more generally be an arbitrary closed region R possessing the sector property described at the end of the preceding section. Let $f(x, y)$ be a function continuous throughout R, let $P_n(x, y)$ be a polynomial of the n-th degree in the two variables together, let

$$G_{ns} = \iint_R | f(x,y) - P_n(x,y) |^s \, dxdy,$$

s being a positive exponent, as before, and let it be supposed that there is a polynomial $p_n(x, y)$, of the n-th degree in the two variables together, such that

$$| f(x,y) - p_n(x,y) | \leqq \epsilon_n$$

at all points of R.

Under these conditions the corresponding form of Markoff's theorem obtained in § 2 is applicable. To give the proof in this single instance in some detail, let

$$f(x,y) - p_n(x,y) = r_n(x,y), \qquad | r_n(x,y) | \leqq \epsilon_n,$$
$$P_n(x,y) - p_n(x,y) = \pi_n(x,y),$$

so that

$$f(x,y) - P_n(x,y) = r_n(x,y) - \pi_n(x,y).$$

Let μ_n be the maximum of $| \pi_n(x,y) |$ in R, taken on at a point (x_0, y_0). At any point (x, y) which belongs to the sector associated with (x_0, y_0) by the

[13] See e. g. D. Jackson, *Bulletin, loc. cit.*, pp. 901-902, Theorems 9, 10.

hypothesis, and is distant from (x_0, y_0) by not more than $h_n = h \sin\gamma/(8n^2)$, since $|\partial\pi_n/\partial s| \leqq 4n^2\mu_n/(h\sin\gamma)$ throughout R and the entire segment joining (x, y) with (x_0, y_0) belongs to R,

$$|\pi_n(x, y) - \pi_n(x_0, y_0)| \leqq \mu_n/2, \qquad |\pi_n(x, y)| \geqq \mu_n/2.$$

If $\mu_n \geqq 4\epsilon_n$, which means that $|r_n(x, y)| \leqq \mu_n/4$,

$$|r_n(x, y) - \pi_n(x, y)| \geqq \mu_n/4$$

at the same points. As this relation holds at least throughout a sector of radius h_n, angle γ, and area $\gamma h_n^2/2$, belonging to R,

$$G_{ns} \geqq \frac{\gamma h_n^2}{2}\left(\frac{\mu_n}{4}\right)^s = \frac{h^2\gamma\sin^2\gamma}{128n^4}\left(\frac{\mu_n}{4}\right)^s, \qquad \mu_n \leqq 4\left(\frac{128n^4 G_{ns}}{h^2\gamma\sin^2\gamma}\right)^{1/s}.$$

Otherwise $\mu_n < 4\epsilon_n$. As μ_n certainly can not exceed the sum of the two alternative upper bounds, while $|\pi_n(x, y)| \leqq \mu_n$ and $|r_n(x, y)| \leqq \epsilon_n$, it is true in either case that

$$(2) \qquad |f(x, y) - P_n(x, y)| = |r_n(x, y) - \pi_n(x, y)| \leqq 4\left(\frac{128n^4 G_{ns}}{h^2\gamma\sin^2\gamma}\right)^{1/s} + 5\epsilon_n$$

throughout R.

Theorems on the existence of approximating polynomials $p_n(x, y)$ with specified upper bounds of error, to be sure, are for approximation over a rectangular region.[14] For direct application of those theorems it is to be assumed that $f(x, y)$ satisfies the requisite conditions of continuity or differentiability on a rectangle containing R, or at least that its definition in R can be so extended that the conditions hold throughout such a rectangle.[15]

In the analogous problem of trigonometric approximation on a region R having the sector property the conclusion corresponding to (2) is that

$$|f(x, y) - T_n(x, y)| \leqq C_s(n^4 G_{ns})^{1/s} + 5\epsilon_n,$$

where C_s, though not so easy to calculate explicitly as in the polynomial case, is again dependent only on h, γ, and s.

Let the phrase "mixed sum" be used to describe a function of x and y which is a polynomial with respect to one variable and a trigonometric sum with respect to the other. Let $f(x, y)$ be a continuous function of the two

[14] See D. Jackson, *Über die Genauigkeit der Annäherung stetiger Funktionen* . . ., Dissertation, Göttingen, 1911, pp. 88-95; E. L. Mickelson, *loc. cit.*, § 5.

[15] In this connection see Hassler Whitney, "Analytic extensions of differentiable functions defined in closed sets," *Transactions of the American Mathematical Society*, vol. 36 (1934), pp. 63-89.

The problem of minimizing G_{ns} can be regarded alternatively as that of minimizing the integral over a rectangle containing R, with a weight function equal to 1 in R and equal to 0 elsewhere.

variables for $a \leqq x \leqq b$ and for all values of y, of period 2π with respect to y, let $U_n(x, y)$ be a mixed sum which is a polynomial of the n-th degree in x and a trigonometric sum of the n-th order in y, and let

$$G_{ns} = \int_{-\pi}^{\pi} \int_{a}^{b} | f(x, y) - U_n(x, y) |^s \, dxdy.$$

Suppose that there exists a mixed sum $u_n(x, y)$, of the n-th degree in x and of the n-th order in y, such that

$$| f(x, y) - u_n(x, y) | \leqq \epsilon_n$$

for $a \leqq x \leqq b$ and for all values of y. By the use of Markoff's theorem (in a single variable) for differentiation with respect to x and of Bernstein's theorem for differentiation with respect to y it is found that

$$| f(x, y) - U_n(x, y) | \leqq 4 \left(\frac{16 n^3 G_{ns}}{b - a} \right)^{1/s} + 5\epsilon_n$$

throughout the strip of the (x, y) plane under consideration, the factor n^3 being associated with G_{ns} here instead of n^2 or n^4. Theorems are available [16] on the existence of approximating sums $u_n(x, y)$ corresponding to specified orders of magnitude of ϵ_n.

A trigonometric substitution can be used in connection with polynomial approximation in the same manner as in the one-dimensional case [17] to improve the order of magnitude obtained for the upper bound of error in the interior of the domain of integration. The substitution can be applied either to one variable or to both. In the hypotheses of an earlier paragraph relating to polynomial approximation over a rectangle, let the rectangle be taken for simplicity as the square $-1 \leqq x \leqq 1$, $-1 \leqq y \leqq 1$. Let N be the smallest integer $\geqq 1/s$. By setting $x = \cos\theta$, $y = \cos\phi$, and using the results obtained above for trigonometric approximation over the entire plane, it may be shown that

$$| f(x, y) - P_n(x, y) | \leqq \frac{4[16(n + N)^2 G_{ns}]^{1/s} + 5\epsilon_n}{[(1 - x^2)(1 - y^2)]^{N/2}}. \tag{3}$$

The significant difference between this and the earlier result for polynomials is the replacement of n^4 by a factor of the order of n^2, with compensating introduction of a denominator which vanishes on the boundary of the square. If the substitution is made for just one variable, say $y = \cos\phi$, and combined with the conclusion of the last paragraph for approximation by mixed sums, it is found that

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[16] E. L. Mickelson, *loc. cit.*, § 6.

[17] See e. g. D. Jackson, *Bulletin*, *loc. cit.*, p. 905, Lemma 8.

$$| f(x,y) - P_n(x,y) | \leqq \frac{4[16(n+N)^3 G_{ns}]^{1/s} + 5\epsilon_n}{(1-y^2)^{N/2}}.$$

This inequality, with a factor of the order of n^3, is effective on two sides of the square as well as in the interior. It appears then, as far as the evidence of the present reasoning goes, that the factor n^4 is required at the corners of the square, a factor of the order of n^3 at points of the sides other than the corners, and a factor of the order of n^2 in the interior. Either of the last two inequalities can be adapted to a rectangle of arbitrary dimensions with sides parallel to the axes by linear transformation of the variables separately. Expressed in terms of the degree of the polynomials in the two variables jointly, they can be carried over by rotation of axes to a rectangle of arbitrary orientation.

For a square of side $2h$ with middle point at (x_0, y_0) the inequality (3), evaluated at the middle point, becomes

$$| f(x_0,y_0) - P_n(x_0,y_0) | \leqq 4[16(n+N)^2 G_{ns}/h^2]^{1/s} + 5\epsilon_n.$$

If the hypotheses are stated for a circle of radius δ and center (x_0, y_0), with G_{ns} now representing the value of the integral over the circle,

$$(4) \qquad | f(x_0,y_0) - P_n(x_0,y_0) | \leqq 4[32(n+N)^2 G_{ns}/\delta^2]^{1/s} + 5\epsilon_n,$$

since the circle contains a square of side $2^{\frac{1}{2}}\delta$ with sides parallel to the axes. If the definition of G_{ns} and the other hypotheses are formulated for an arbitrary closed region R, it follows that the right-hand member of (4) is an upper bound for $| f(x,y) - P_n(x,y) |$ at any point of R whose minimum distance from the boundary is $\geqq \delta$. For the dependence of the quantity in brackets on n a factor of the order of n^2 is thus sufficient at any interior point of R. The significance of this observation for the theory of convergence of polynomials of closest approximation is brought out more explicitly by the following inference from it:

If $f(x,y)$ is continuous throughout a closed region R, and if $P_n(x,y)$ is determined among all polynomials of the n-th degree in each variable so as to minimize the integral

$$G_{ns} = \iint_R | f(x,y) - P_n(x,y) |^s \, dxdy,$$

$P_n(x,y)$ will converge toward $f(x,y)$ at every interior point of R, uniformly throughout any closed region interior to R, as n becomes infinite, if there exist polynomials $p_n(x,y)$ of corresponding degree such that

$$| f(x,y) - p_n(x,y) | \leqq \epsilon_n$$

throughout R, with $\lim_{n\to\infty} n^{2/s}\epsilon_n = 0$.

This statement, like others of similar character, can be generalized by introduction of a weight function.

For an arbitrary interior point of an arbitrary rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, with sides parallel to the axes, (3) takes the form

$$| f(x, y) - P_n(x, y) | \leqq \frac{C_s[(n^2 G_{ns})^{1/s} + \epsilon_n]}{[(b-x)(x-a)(d-y)(y-c)]^{N/2}}$$

where C_s depends only on s and on the dimensions of the rectangle. By successive applications, after the manner of a corresponding proof in one dimension,[18] this can be made to yield a theorem of similar form with regard to trigonometric approximation. The four overlapping intervals of the argument in one variable are to be replaced here by sixteen overlapping rectangles. It is to be supposed that $f(x, y)$ is continuous throughout a rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, where $0 < b - a < 2\pi$, $0 < d - c < 2\pi$, and that there exists a trigonometric sum $t_n(x, y)$ of the n-th order in each variable such that $| f(x, y) - t_n(x, y) | \leqq \epsilon_n$ throughout the rectangle. Then if $T_n(x, y)$ is an arbitrary trigonometric sum of similar order, if s is an arbitrary positive exponent, if

$$G_{ns} = \int_c^d \int_a^b | f(x, y) - T_n(x, y) |^s \, dx dy,$$

and if N is the smallest integer $\geqq 1/s$, the conclusion is that

$$| f(x, y) - T_n(x, y) | \leqq \frac{C_s[(n^2 G_{ns})^{1/s} + \epsilon_n]}{[(b-x)(x-a)(d-y)(y-c)]^{N/2}}$$

at all interior points, with a new C_s which again depends only on s and on the dimensions of the rectangle.

As in the discussion of polynomial approximation just preceding, this result can be interpreted in particular for the middle point of a square, then for the center of a circle, and so ultimately for the interior of an arbitrary closed region.

It is clear that the conclusions that have been formulated above are merely illustrative of a very large number which could be obtained by similar methods.

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[18] D. Jackson, *Transactions, loc. cit.*, pp. 246-248, Lemma 6.

ON MULTIPARAMETER EXPANSIONS ASSOCIATED WITH A DIFFERENTIAL SYSTEM AND AUXILIARY CONDITIONS AT SEVERAL POINTS IN EACH VARIABLE.*

By Chester C. Camp.

1. Introduction. In 1916 C. E. Wilder presented problems in the theory of ordinary differential equations with auxiliary conditions at more than two points.[1] It is the purpose of this paper to extend the theory in two directions.

First by starting with the partial differential equation

$$\sum_{a=1}^{p} L_a(u) + \lambda u = 0 \tag{1}$$

and the auxiliary conditions

$$U_{a\beta}(u) = 0, \tag{2}$$

one is led by Bernoulli's method of solution to what may be called a multiple Wilder system in terms of which one may expand a function $f(x_1, x_2, \cdots, x_p)$ in a multiple Wilder series. Here the coefficient of each new parameter λ_a is unity as in Wilder's differential equation the coefficient of λ is one.

Secondly is considered the system

$$X'_j + [\sum_{i=1}^{p} \lambda_i a_{ji}(x_j)] X_j = 0, \qquad (j = 1, 2, \cdots, p); \tag{3}$$

with the auxiliary conditions

$$X_j(a_j) + c_{j2}X_j(a_{2j}) + \cdots + c_{j,k_j-1}X_j(a_{k_j-1,j}) + c_{j,k_j}X_j(b_j) = 0, \tag{4}$$

where $c_{j,k_j} \neq 0$, $(j = 1, 2, \cdots, p)$.

Here the coefficients of the parameters λ_i will be assumed not to change sign but to maintain their average values in each subinterval. A properly restricted function f will then be expansible in a series of characteristic solutions which will converge almost everywhere to f.

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* Presented to the American Mathematical Society, in a different form, August 29, 1929. Received by the Editors, August 17, 1937.

[1] *Transactions of the American Mathematical Society*, vol. 18, pp. 415-442 and vol. 19, pp. 157-166.

The system (3) may also be extended to the more general form

$$(3.1)\qquad X'_j + [\sum_{i=1}^{p} \lambda_i a_{ji}(x_j) + g_{ji}(x_j)]X_j = 0, \qquad (j = 1, 2, \cdots, p);$$

which is considerably more general than the system considered in a previous paper by the author.[2]

2. Multiple Wilder system. As in the author's earlier work on Multiple Birkhoff Series[3] one is led from (1), (2) by a simple transformation to a system of p ordinary differential equations of order n_α, $(\alpha = 1, 2, \cdots, p)$ and p boundary systems

$$(5)\qquad L_\alpha(u_\alpha) + \lambda_\alpha u_\alpha = 0, \qquad W_{\alpha\beta}(u_\alpha) = 0, \qquad (\beta = 1, 2, \cdots, n_\alpha).$$

The boundary conditions (2) can be so formulated that, for a particular α, (5) will constitute a Wilder system,[4] where auxiliary conditions are imposed at $k_\alpha > 2$ points. By proceeding as in the article referred to, one may set up multiple series in terms of principal solutions of (5) directly, without reference to (1), (2). In extending the convergence proof from the case of one to p independent variables one may employ the extension of the contour integral method.[5] Although the work is a more arduous piece of routine it can be accomplished in a similar manner[6] without unforeseen difficulties. One may therefore state

THEOREM I. *Given $f(x_1, x_2, \cdots, x_p)$, any real function, which together with its first ν_α*[7] *partial derivatives with respect to x_α (or if $\nu_\alpha < 1$, with its first partial derivative) is continuous in the region (a_α, b_α), $(\alpha = 1, 2, \cdots, p)$; then the multiple Wilder expansion converges to f at every interior point of this region, if the auxiliary conditions are such that for each α certain determinants of the matrix of constants do not vanish and if the $a_{\alpha\beta}$ are such that the*

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[2] Camp, "An expansion involving p inseparable parameters associated with a partial differential equation," *American Journal of Mathematics*, vol. 50, p. 259, equations (5).

[3] Camp, "Expansions in terms of solutions of partial differential equations," Second Paper, Multiple Birkhoff Series, *Transactions of the American Mathematical Society*, vol. 25, pp. 338-342.

[4] See Wilder's Theorem for restrictions on certain constants in $W_{\alpha\beta}$, Wilder, *loc. cit.*, vol. 18, p. 433.

[5] Camp, Multiple Fourier Series, *Transactions of the American Mathematical Society*, vol. 25, pp. 131, 132.

[6] Cf. the three articles by the author cited above.

[7] See the statement of Wilder's theorem, *loc. cit.*, p. 433.

intervals $(a_{2\alpha}, b_\alpha)$ *and* $(a_\alpha, a_{k_\alpha - 1})$ *are longer than any of the other subintervals of* (a_α, b_α) *included between any two of the points* $a_{\alpha\beta}$, $(\beta = 1, 2, \cdots, k_\alpha)$.

3. Generalized first order system. Consider the system (3), (4). If one stipulates that $X_j(x_j)$ shall be continuous, then for each j one may determine a system of conjugate auxiliary conditions by the use of the integral of Lagrange's identity [8] allowing $Y_j(x_j)$, the solution of the system of adjoint differential equations,

$$(6) \qquad -Y'_j + [\sum_{i=1}^{p} \lambda_i a_{ji}(x_j)]Y_j = 0, \qquad (j = 1, 2, \cdots, p);$$

to be discontinuous at each interior auxiliary point a_{ij}, $(i = 2, 3, \cdots, k_j - 1)$. This system as a whole will be unique; it can be written in various forms, one of which is the following:

$$(7) \qquad \begin{cases} c_{j,k_j} Y_j(a_j) + Y_j(b_j) = 0 \\ c_{ji} Y_j(a_j) + Y_j(a_{ij}^-) - Y_j(a_{ij}^+) = 0, & (i = 2, 3, \cdots, k_j - 1). \end{cases}$$

Write a solution of (3) in the form

$$(8) \qquad X_j(x_j) = 1/\exp \sum_{i=1}^{p} \lambda_i A_{ji}(x_j) \quad \text{where } A_{ji}(x_j) = \int_{a_j}^{x_j} a_{ji}(x_j)\, dx_j.$$

A solution of (6) may be written

$$(9) \qquad Y_j(x_j) = K_{ij} \exp \sum_{i=1}^{p} \lambda_i A_{ji}(x_j), \qquad a_{ij} < x_j < a_{i+1,j},$$

$(i = 1, 2, \cdots, k_j - 1)$; where a_{1j} is defined as a_j, and a_{k_j} as b_j. If one takes arbitrarily $K_{1j} = 1$, then the conditions (7) will determine the other K's in succession. The first equation in (7) will determine $K_{k_j - 1, j}$ independently, which will be consistent with the other determination provided equation (4) is satisfied. Clearly then the principal parameter values for both systems are the same if they exist. In order to investigate their existence it is expedient to make the following transformations:

$$(10) \quad \nu_j = \sum_{i=1}^{p} \lambda_i A_{ji} \text{ where } A_{ji} = \int_{a_j}^{b_j} a_{ji}(x_j)\, dx_j / (b_j - a_j), \; (j = 1, 2, \cdots, p).$$

The new parameter ν_j is the average value of the coefficient of X_j in (3) over the interval (a_j, b_j). In order to solve backwards we must now assume that the determinant $| A_{ji} |$ does not vanish. If in addition we assume that each $a_{ji}(x_j)$ maintains its average value over each subinterval; i. e.,

[8] Wilder, *loc. cit.*, vol. 19, p. 162.

$$(11)\qquad \int_{a_{ij}}^{a_{i+1,j}} a_{jk}(x_j)dx_j/(a_{i+1,j}-a_{ij}) = A_{jk},$$
$$(i=1,2,3,\cdots,k_j-1;\ j=1,2,\cdots,p;\ k=1,2,\cdots,p);$$

then (4) in view of (8) becomes

$$(12)\qquad W_j(\nu_j) \equiv 1+\sum_{i=2}^{k_j} c_{ji}\, e^{-(a_{ij}-a_j)\nu_j} = 0, \qquad (j=1,2,\cdots,p).$$

If $a_{ji}(x_j)$ is real and integrable then by a theorem given by Langer[9] and due to Wilder and Tamarkin the solutions of (12) lie within a strip $|R(\nu_j)| < K$ and in any rectangular portion of this strip the number of roots $n(R)$ is limited by the relation

$$-n + C(y_2-y_1) \leqq n(R) \leqq n + C(y_2-y_1)$$

where y_1, y_2 are two values of the imaginary part of ν_j, $R(\nu_j)$ denotes its real part, and C is the absolute value of the numerically largest coefficient of ν_j in (12).

In case the subintervals are commensurable with the whole interval (a_j, b_j), $W_j(\nu_j)$ reduces to a polynomial of degree p_n in $\exp \alpha\nu_j$ and its zeros are of the form

$$(13)\qquad \nu_j = [2m\pi i + \log \xi_k]/\alpha, \quad (k=1,2,\cdots,p_n;\ m=0,\pm 1,\pm 2,\cdots),$$

provided the zeros occur at points for which $\exp \alpha\nu_j = \xi_k$.[10]

In either case when ν_j is uniformly bounded from the roots of (12) then $W_j(\nu_j)$ is uniformly bounded from zero.

The Green's system may be written

$$(14)\quad G \equiv \begin{cases} \prod_{j=1}^{p} [X_j(x_j)Y_j(s_j)/W_j(\nu_j)(b_j-a_j)]/|A_{ji}|, \ s_j < x_j, \\ \hfill (j=1,2,\cdots,p); \\ \prod_{j=1}^{p} [X_j(x_j)Y_j(s_j)\{1-W_j(\nu_j)\}/W_j(\nu_j)(b_j-a_j)]/|A_{ji}|, \ s_j > x_j, \\ \hfill (j=1,2,\cdots,p); \end{cases}$$

where $Y_j(s_j)$ is the discontinuous solution defined by (9), and where the individual factor $1/W_i(\nu_i)$ in the first form of G in (14) is to be replaced by $\{1-W_i(\nu_i)\}/W_i(\nu_i)$ for every independent variable s_i whenever $s_i > x_i$.

If one wishes to expand a function $f(x_1, x_2, \cdots, x_p)$ as

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[9] R. E. Langer, "On the zeros of exponential sums and integrals," *Bulletin of the American Mathematical Society*, vol. 37, pp. 218-219. See his references to Wilder and Tamarkin on p. 239.

[10] Cf. Langer, *loc. cit.*, pp. 214-215.

$$(15) \qquad f(x_1, x_2, \cdots, x_p) = \sum_{m_j=-\infty}^{\infty} C_{m_1, m_2, \ldots, m_p} \prod X^*_j(x_j) \qquad (j = 1, 2, \cdots, p);$$

where the asterisk denotes a principal solution, then

$$(16) \quad C_{m_1, m_2, \ldots, m_p} = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_p}^{b_p} | a_{ji}(s_j) | f(s_1, s_2, \cdots, s_p) \prod_{j=1}^{p} Y^*_j(s_j) ds_j / Q$$

where

$$(17) \qquad Q = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_p}^{b_p} | a_{ji}(s_j) | \prod_{j=1}^{n} X^*_j(s_j) Y^*_j(s_j) ds_j.$$

If a double star indicates a distinct set of principal parameter values, one has the conjugacy condition

$$(18) \qquad \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_p}^{b_p} | a_{ji}(s_j) | \prod_{j=1}^{n} X^*_j(s_j) Y^{**}_j(s_j) ds_j = 0.$$

Moreover the residue at a simple set of principal parameter values ν^*_j of the function

$$(19) \qquad \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_p}^{b_p} f(s_1, s_2, \cdots, s_p) \; | a_{ji}(s_j) | \; G \prod_{j=1}^{p} ds_j$$

will give the corresponding term of (15) since Q can be shown to take the form

$$(20) \qquad Q = | A_{ji} | \prod_{j=1}^{p} (b_j - a_j) W'_j(\nu^*_j).$$

The convergence proof for the series in (15) is made by the extended contour integral method [11] and the use of the following lemmas.

Lemma I.

$$\lim_{|z| \to \infty} \frac{1}{2\pi i} \int_{C_z} \frac{1}{W_j(z)} \frac{dz}{z} = \tfrac{1}{2}, \qquad c_{j,k_j} \neq 0, \qquad (j = 1, 2, \cdots, p);$$

Lemma II.

$$\lim_{|z| \to \infty} \frac{1}{2\pi i} \int_{C_z} \frac{| e^{-hz} |}{W_j(z)} \frac{dz}{z} = 0, \qquad 0 < h < b_j - a_j;$$

where C_z is a circular contour with center at $z = 0$ uniformly bounded away from zeros of $W_j(z)$. In case $W_j(z)$ contains the factors $1 - e^{\alpha z}$ and $1 - e^{\beta z}$ where α and β are incommensurable it may be necessary to use a sequence of circles C_z of non-uniformly increasing radii as $| z | \to \infty$. The corresponding

[11] Cf. Camp, *American Journal of Mathematics, loc. cit.*, pp. 262 sqq.

expansion series will be arranged accordingly in groups of characteristic solutions in order that it may converge properly. With this proviso one may enunciate the

THEOREM II. *Let* $f(x_1, x_2, \cdots, x_p)$ *be made up of a finite number of pieces, each real and possessing a continuous partial derivative in each argument in the region* S: $a_j \leqq x_j \leqq b_j$, $(j = 1, 2, \cdots, p)$; *let each* $a_{ji}(x_j)$ *be integrable and either identically zero or of constant sign* $a_j \leqq x_j \leqq b_j$; *let the average value of* $a_{ji}(x_j)$ *over the subinterval* $a_{ij} \leqq x_j \leqq a_{i+1,j}$ *be* A_{ji} *the same for all the subintervals; and let the determinant* $| A_{ji} |$ *be different from zero. Then the expansion* (15) *will converge at any interior point of* S *to the so-called mean value of* f. *If the terms of* (15) *are grouped appropriately the series will converge uniformly to* f *at an interior point of* S *at which* f *is continuous. In the case of a multiple characteristic value it is to be understood that the corresponding term of* (15) *is to be replaced by*

$$(21) \qquad \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_p}^{b_p} f(s_1, s_2, \cdots, s_p)\,|\,a_{ji}(s_j)\,|\,R^*\prod ds_j$$

where R^* *is the residue of the Green's function in* (14).

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CONCERNING SOME POLYNOMIALS ORTHOGONAL ON A FINITE OR ENUMERABLE SET OF POINTS.*

By Morris J. Gottlieb.

1. Introduction. Let $x_0 < x_1 < x_2 < \cdots$ be a monotonically increasing sequence of points on the real axis, and $j_0, j_1, j_2, \cdots$ a sequence of positive numbers. Then a system of polynomials $\{p_n(x)\}$ can be defined such that $p_n(x)$ is of the exact degree n, and the orthogonality relations

$$(1.1) \qquad \sum_{\nu=0}^{\infty} j_\nu p_n(x_\nu) p_m(x_\nu) = \begin{cases} 0, n \neq m, \\ 1, n = m, \end{cases} \qquad (n, m = 0, 1, 2, \cdots)$$

hold. These polynomials are uniquely determined except for a factor of ± 1. This follows from the general theory of orthogonal polynomials.

In the "finite" case, in which only a finite number of points $x_0 < x_1 < \cdots < x_N$, and positive values, $j_0, j_1, \cdots, j_N$ are given, a finite system, $p_0(x), p_1(x), \cdots, p_N(x)$, can be defined with the analogous property.

Some special cases occur in the literature. We mention the following:

(a) Tchebychef [10] investigates in detail the "finite" case with $j_0 = j_1 = \cdots = j_N = 1$.

(b) Tchebychef [11] and Ch. Jordan [3] consider the "finite" case of equidistant points with $j_0 = j_1 = \cdots = j_N = 1$. The corresponding polynomials represent a finite analogue of Legendre polynomials. They go over by a proper limiting process into Legendre polynomials.

(c) Krawtchouk [4] considers the finite case, with equidistant points, say $x_\nu = \nu$, and $j_\nu = \binom{N}{\nu} p^\nu q^{N-\nu}$, $(\nu = 0, 1, 2, \cdots, N)$. Here $p > 0$, $q > 0$, $p + q = 1$. He obtains a set of polynomials, which by a suitable limiting process, go over into the Hermite polynomials.

(d) A well known case is that of the Poisson-Charlier polynomials first investigated by Charlier [1]. Here, we have a sequence of equidistant points $x^\nu = \nu$, and $j_\nu = e^{-a} \dfrac{a^\nu}{\nu!}$, $(\nu = 0, 1, 2, \cdots)$, where $a > 0$.

(e) In this paper, we are concerned with the case of equidistant points, $x_\nu = \nu$ and $j_\nu = e^{-\lambda\nu}$, $(\nu = 0, 1, 2, \cdots; \lambda > 0)$. The relation of the corresponding polynomials to the Laguerre polynomials is similar to that in case (b) to the Legendre polynomials. An indication of this distribution is given by Stieltjes [9], in his classical paper on continued fractions. However, he does not seem to devote any further discussion to these polynomials.

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The derivation of the usual formal properties of the polynomials (e) presents no difficulties. We mention these formulae in Section 2 without proof and turn our attention in Sections 3, 4 and 5 to the questions of the asymptotic behavior for polynomials of large degree and of the development problem.

In Section 6 some remarks on the cases (c) and (d) are also made.

2. Polynomials of the "Laguerre type." Definition and formal properties. We can show by Abel's transformation (summation by parts) that the polynomials, $l_n(x)$, defined by the formula

$$e^{-\lambda x} l_n(x) = \Delta^n \left\{ e^{-\lambda x} \binom{x}{n} \right\}, \qquad (\lambda > 0), \tag{2.1}$$

satisfy the orthogonality and normalization relations:

$$\sum_{\nu=0}^{\infty} e^{-\lambda\nu} l_n(\nu) l_m(\nu) = \begin{cases} 0, & n \neq m, \\ e^{-n\lambda}(1 - e^{-\lambda})^{-1}, & n = m, \end{cases} \quad (n, m = 0, 1, 2, \cdots). \tag{2.2}$$

This is exactly the case mentioned in § 1, (e).

The highest coefficient in $l_n(x)$ is of the sign $(-1)^n$. We notice that $\sum_{\nu=0}^{\infty} j_\nu = \sum_{\nu=0}^{\infty} e^{-\lambda\nu} = (1 - e^{-\lambda})^{-1}$.

By use of Newton's series, we obtain

$$l_n(x) = e^{-n\lambda} \sum_{\nu=0}^{n} (1 - e^{\lambda})^{\nu} \binom{n}{\nu} \binom{x}{\nu}. \tag{2.3}$$

The polynomials $l_n(x)$, may be represented by certain Jacobi polynomials. Using the notation of Pólya-Szegö [6, vol. 2, p. 93] we find

$$l_n(x) = P_n^{(0, x-n)}(2e^{-\lambda} - 1). \tag{2.4}$$

If $x = 0, 1, 2, \cdots$, a symmetry property analogous to that of the Poisson-Charlier polynomials [1] is observed from (2.3); namely,

$$e^{n\lambda} l_n(x) = e^{x\lambda} l_x(n), \qquad (x = 0, 1, 2, \cdots). \tag{2.5}$$

In the usual way, the recurrence formula

$$\begin{aligned} (n+1) l_{n+1}(x) - \{(n+1)e^{-\lambda} + n + (e^{-\lambda} - 1)x\} l_n(x) \\ + n e^{-\lambda} l_{n-1}(x) = 0 \end{aligned} \tag{2.6}$$

follows. This is valid for $n = 0, 1, 2, \cdots$, with arbitrary definition of $l_{-1}(x)$.

We notice the following difference equation:

$$\begin{aligned} & e^{-\lambda}(x+2)\Delta^2 l_n(x) \\ & \quad - \{(1 - e^{-\lambda})x + (n-2)e^{-\lambda} - (n-1)\}\Delta l_n(x) \\ & \quad + n(1 - e^{-\lambda}) l_n(x) = 0. \end{aligned} \tag{2.7}$$

For later purposes, the generating function

[1] See, for instance, [2].

$$G(x, w) = \sum_{n=0}^{\infty} l_n(x) w^n = (1-w)^x (1-e^{-\lambda} w)^{-x-1}, \quad (|w| < 1) \tag{2.8}$$

is important.

The polynomials, $l_n(x) \equiv l_n(x;\lambda)$ are connected with the Laguerre polynomials by the following limiting relation:

$$\lim_{\lambda \to 0} l_n\left(\frac{x}{\lambda};\lambda\right) = L_n(x), \tag{2.9}$$

where for the Laguerre polynomials, the notation of Pólya-Szegö [6, vol. 2, p. 94] has been used. This follows immediately from (2.1).

3. Asymptotic formula. By use of the classical method of Darboux we can readily derive an asymptotic formula for $l_n(x)$ which is uniformly valid for large n in a fixed bounded region of the complex x-plane.

The second factor $\phi(w) = (1-e^{-\lambda}w)^{-x-1}$ of the generating function (2.8) is regular in $|w| < e^{\lambda}$; it may be developed in a Taylor's series around $w = 1$, so that

$$\begin{cases} G(x, w) = \sum_{\nu=0}^{k} \phi_\nu (1-w)^{x+\nu} + \sum_{\nu=k+1}^{\infty} \phi_\nu (1-w)^{x+\nu} \\ \qquad = S_k(w) + r(w). \end{cases} \tag{3.1}$$

Here k is a fixed integer, and $\phi_\nu = (-1)^\nu \dfrac{\phi^{(\nu)}(1)}{\nu!}$.

Let now $|x| \leqq \omega$, $(\omega > 0)$; we choose $k > \omega$. Then $r(w)$ is bounded with its first $[k+1-\omega]$ derivatives in $|w| \leqq 1$. Hence, if we write

$$r(w) = \sum_{n=0}^{\infty} c_n w^n, \tag{3.2}$$

we have

$$c_n = O(n^{-[k+1-\omega]}). \tag{3.3}$$

If now, k is chosen so that $k > 2\omega + 2$ we find [2]

$$l_n(x) = (-1)^n (1-e^{-\lambda})^{-x-1} \binom{x}{n} + O\{n^{-R(x)-2}\} \tag{3.4}$$

uniformly in $|x| \leqq \omega$.

The same method leads readily to a complete asymptotic expansion of $l_n(x)$.

4. Location of the zeros. The asymptotic formula (3.4) yields some information concerning the location of the zeros of $l_n(x)$ if n is large. From the general theory of orthogonal polynomials it follows that the zeros of $l_n(x)$ lie on the positive, real axis; more exactly, there are exactly n open intervals of the form, $\nu < x < \nu + 1$, $\nu = 0, 1, 2, \cdots$, each containing one zero in its interior.[3]

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[2] Here, $R(x)$ denotes the real part of x.

[3] This is generally true in the case of the polynomials defined by (1.1), replacing ν by x_ν. See [7].

From the asymptotic formula (3.4) it is apparent that for any fixed positive integer ν, and an arbitrarily small δ, $0 < \delta < \frac{1}{2}$, an $n(\delta)$ may be found, so that for $n > n(\delta)$, $l_n(x)$ has an odd number of zeros in $\nu - \delta < x < \nu + \delta$. Because of the property mentioned before, there is, then, *exactly one* zero in this interval.

5. Development theorem. E. Schmidt [8] has obtained very complete results on the development of an arbitrary function (defined for non-negative integer values of the argument) in terms of the Poisson-Charlier polynomials. His method can be taken over with slight modifications to discuss the analogous question for our polynomials.

Let $f(x)$ be an arbitrary function defined for $x = 0, 1, 2, \cdots$. We discuss the "Fourier development"

$$f(x) \sim \sum_{n=0}^{\infty} c_n l_n(x) \tag{5.1}$$

where the "Fourier constants," c_n, are defined by

$$c_n = e^{n\lambda}(1 - e^{-\lambda}) \sum_{x=0}^{\infty} e^{-\lambda x} f(x) l_n(x). \quad \text{(See 2.2.)} \tag{5.2}$$

The question is: Under what conditions, regarding $f(x)$, do the expressions c_n have a sense and does the series (51) converge and represent $f(x)$? The first requirement means that the series (5.2) is convergent for each n, $n = 0, 1, 2, \cdots$.

The answer can be stated as follows: A sufficient condition that the development (5.1) exists (i.e., that the constants c_n have a sense) is that the analytic function

$$F(z) = \sum_{\nu=0}^{\infty} f(\nu) z^{\nu}$$

be regular in $|z| < e^{-\lambda}$, and in $\left| z - \dfrac{1}{1+e^{\lambda}} \right| < \dfrac{1}{1+e^{\lambda}}$, and have bounded derivatives of all orders interior to both circles. If $F(z)$ is regular at $z = e^{-\lambda}$, then this condition is also necessary.

The proof follows closely the discussion of E. Schmidt. The essential idea is the use of the symmetry property (2.5), which is analogous to a similar property of the Poisson-Charlier polynomials.

6. Some remarks concerning Poisson-Charlier polynomials and the polynomials of Krawtchouk.

(a) The Poisson-Charlier polynomials [case (d) of § 1] can be represented in the form

$$p_n(x) = \sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}\nu!\,a^{-\nu}\binom{x}{\nu}, \qquad (a > 0) \tag{6.1}$$

cf. Doetsch [2]; they have the following generating function, [2]:

$$H(x,w) = \sum_{n=0}^{\infty} p_n(x)\frac{w^n}{n!} = e^{-w}\left(1+\frac{w}{a}\right)^x, \qquad (|w| < a). \tag{6.2}$$

By Darboux's method, the following asymptotic formula for $p_n(x), n \to \infty$, valid uniformly in a fixed region of the complex x-plane is readily obtained from the generating function [4]

$$p_n(x) = e^a a^{-n} n!\binom{x}{n} + a^{-n}n!\,O(n^{-R(x)-2}). \tag{6.3}$$

As in the case of the polynomials $\{l_n(x)\}$ treated above, it follows that for sufficiently large n, $p_n(x)$ has exactly one zero in each interval $\nu - \delta < x < \nu + \delta$ where ν is fixed, $\nu = 0, 1, 2, \cdots$, and δ is arbitrarily small with $0 < \delta < \frac{1}{2}$.

(b) The Krawtchouk polynomials have the following explicit representation, [4]

$$k_n(x) = \sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{N-x}{n-\nu}\binom{x}{\nu}p^{n-\nu}q^{\nu}. \tag{6.4}$$

Here, $p > 0$, $q > 0$, and $p + q = 1$. The polynomials defined by this formula have a sense also for $n > N$, although they then cease to satisfy the normalization conditions, since they vanish for $x = 0, 1, 2, \cdots, N$.

For these polynomials also, an asymptotic formula can be obtained by Darboux's method, valid uniformly for large n, in a fixed region of the complex x-plane. We start out from the generating function, [4],

$$K(x,w) = \sum_{n=0}^{\infty} k_n(x)w^n = (1+qw)^x(1-pw)^{N-x}, \quad \left[\,|w| < \min\left(\frac{1}{p},\frac{1}{q}\right)\right]. \tag{6.5}$$

It is necessary to distinguish three different cases according to different values of the constants p and q. In case $p < q$, the asymptotic formula is

$$k_n(x) = q^n\left\{q^{-(N-x)}\binom{x}{n} + O(n^{-R(x)-2})\right\}. \tag{6.6}$$

By interchanging p and q, x and $N - x$, w and $-w$, the asymptotic formula for $p > q$ is obtained from the previous case:

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[4] Doetsch [2] mentions the possibility of obtaining the asymptotic formula in this way, without discussing it further. Cf. also the preliminary communication of Obrechkoff [5].

$$(6.7) \qquad k_n(x) = p^n \left\{ (-1)^n p^{-x} \binom{N-x}{n} + O(n^{-[N-R(x)]-2}) \right\}.$$

In the case $p = q = \frac{1}{2}$, the asymptotic formula obtained by Darboux's method is

$$(6.8) \qquad k_n(x) = 2^{-n} \left\{ 2^{N-x} \binom{x}{n} + (-1)^n 2^x \binom{N-x}{n} + O(n^{-\xi-2}) \right\},$$

where $\xi = \min \{R(x), N - R(x)\}$.

All three formulae are valid uniformly in a fixed region of the complex x-plane. The same method leads readily to a complete asymptotic expansion of $k_n(x)$.

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ON THE FOURIER-STIELTJES TRANSFORM OF A SINGULAR FUNCTION.*

By Philip Hartman and Richard Kershner.

It is a well known consequence of the Riemann-Lebesgue lemma that the Fourier-Stieltjes transform,

$$L(t;\sigma)=\int_{-\infty}^{\infty}\exp(itx)\,d\sigma(x), \tag{1}$$

of an absolutely continuous distribution function $\sigma(x)$ satisfies

$$L(t;\sigma)=o(1), \qquad t\to\pm\infty. \tag{2}$$

On the other hand, if $\sigma(x)$ is any distribution function for which

$$L(t;\sigma)=O(|t|^{-1-\epsilon}), \qquad t\to\pm\infty,$$

for some $\epsilon>0$, then $\sigma(x)$ is necessarily absolutely continuous.

If $\sigma(x)$ is a purely singular distribution function, then (2) need not hold.[1] The first example to the effect that (2) may hold when t runs through integral multiples of 2π, even when $\sigma(x)$ is purely singular, was given by Menchoff [2] in the analogous case of Fourier-Stieltjes coefficients of a periodic function. A somewhat simpler example of a purely singular distribution function $\sigma(x)$ for which not only (2) held but

$$L(t;\sigma)=O(\log^{-\gamma}|t|), \qquad t\to\pm\infty \tag{3}$$

for a certain positive γ, was given by one of the authors.[3] More recently, Littlewood [4] has given a very complicated example of a purely singular distribution function $\sigma(x)$ for which

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[1] An example to this effect is the Cantor function.

[2] D. Menchoff, "Sur l'unicité du développement trigonométrique," *Comptes Rendus*, vol. 163 (1916), pp. 433-436.

[3] R. Kershner, "On singular Fourier-Stieltjes transforms," *American Journal of Mathematics*, vol. 58 (1936), pp. 450-452.

[4] J. E. Littlewood, "On the Fourier coefficients of functions of bounded variation," *Quarterly Journal of Mathematics, Oxford Series*, vol. 7 (1936), pp. 219-226. Actually, Littlewood constructed a function $f(x)$ of period 1, which is continuous for all x and of bounded variation in $0\leqq x\leqq 1$, such that if a_n, $n=0,\pm 1,\cdots$ are the Fourier coefficients of $f(x)$, then $a_n=O(|n|^{-1-c})$. This function was of the type $f(x)$

$$(4) \qquad L(2\pi n;\sigma) = O(|n|^{-c}), \qquad n \to \pm\infty \qquad (n = 0, \pm 1, \cdots)$$

for some positive c. In all the above cases, the distribution functions treated were not only purely singular but, in fact, almost everywhere constant, i. e., in the language of distribution functions, their spectra were zero sets.

The object of the present note is to show that (2) and, in fact, (3) may hold in the case of a purely singular distribution function the spectrum of which is an interval. It may be mentioned that, for the example to be given, it is certain that (4) does not hold.

Let $\sigma_n = \sigma_n(x)$ denote the distribution function

$$\begin{aligned} \sigma_n(x) &= 0; & -\infty < x \leqq 0; & \\ \sigma_n(x) &= \tfrac{1}{2}(1 + n^{-\frac{1}{2}}); & 0 < x \leqq (\tfrac{1}{2})^n; & \qquad (n = 1, 2, \cdots); \\ \sigma_n(x) &= 1; & (\tfrac{1}{2})^n < x < +\infty. & \end{aligned}$$

Let $\tau(x)$ denote the infinite convolution

$$\tau(x) = \sigma_1 * \sigma_2 * \cdots.$$

This Poisson convolution is absolutely convergent[5] and represents a continuous[6] purely singular[6] distribution function. Furthermore, the spectrum of $\tau(x)$ is the interval $0 \leqq x \leqq 1$, so that in this interval $\tau(x)$ is strictly increasing.

With the above notations it will be shown that the *purely singular distribution function $\tau(x)$, whose spectrum is an interval, has a Fourier transform $L(t;\tau)$ which satisfies*

$$(7) \qquad L(t;\tau) = O(\log^{-\frac{1}{2}}|t|), \qquad t \to \pm\infty$$

and also

$$(8) \qquad L(t;\tau) = \Omega(\log^{-\frac{1}{2}}|t|), \qquad t \to \pm\infty.$$

Obviously,

$$(9) \qquad L(t;\sigma_n) = \tfrac{1}{2}(1 + n^{-\frac{1}{2}}) + \tfrac{1}{2}(1 - n^{-\frac{1}{2}})\exp(i2^{-n}t).$$

Hence, by the multiplication rule of Fourier transforms,

$$(10) \qquad L(t;\tau) = \prod_{n=1}^{\infty} [\tfrac{1}{2}(1 + n^{-\frac{1}{2}}) + \tfrac{1}{2}(1 - n^{-\frac{1}{2}})\exp(i2^{-n}t)],$$

$= \sigma(x) - x$, $0 \leqq x \leqq 1$, where $\sigma(x)$ is a purely singular monotone function such that $\sigma(0) = 0$, $\sigma(1) = 1$. Now, $\sigma(x)$ may be considered to be a distribution function if one places $\sigma(x) = 0$ for $-\infty < x \leqq 0$ and $\sigma(x) = 1$ for $1 \leqq x < +\infty$. It is clear that $a_n = (2\pi i n)^{-1}L(-2\pi n;\sigma)$.

[5] B. Jessen and A. Wintner, "Distribution functions and the Riemann zeta function," *Transactions of the American Mathematical Society*, vol. 38 (1935), pp. 48-88.

[6] P. Hartman and R. Kershner, "On the structure of monotone functions," *American Journal of Mathematics*, vol. 59 (1937), pp. 809-822.

so that

$$| L(t;\tau)|^2 = \prod_{n=1}^{\infty} [\tfrac{1}{2}(1+n^{-1}) + \tfrac{1}{2}(1-n^{-1}) \cos 2^{-n}t]$$

or

$$(11) \qquad | L(t;\tau)|^2 = \prod_{n=1}^{\infty} [n^{-1} \sin^2 2^{-n-1}t + \cos^2 2^{-n-1}t].$$

Now let λ denote the set of all points $t > 0$ which are within a distance $\pi/6$ of an integral multiple of π, so that there exists a positive $\delta < 1$ such that

$$(12) \qquad | n^{-1} \sin^2 t + \cos^2 t | < \delta < 1, \ n > 1, \text{ if } t \not\subset \lambda.$$

Also let μ_n denote the set of all points $t > 0$ which are within a distance $2^{-n}\pi/6$ of an odd multiple of $\pi/2$, so that

$$(13) \qquad | \cos t | \leqq 2^{-n-1}, \text{ if } t \subset \mu_n.$$

Now let $t \geqq 2\pi$ be fixed and let m ($\geqq 1$) be the unique integer such that

$$(14) \qquad 2^m\pi \leqq t < 2^{m+1}\pi.$$

Suppose that k $(0 \leqq k \leqq m)$ of the m values $1, 2, \cdots, m$ of n are such that $2^{-n-1}t$ is in λ. Suppose that these k values of n consist of j $(1 \leqq j \leqq k)$ sets g_i, $(i = 1, 2, \cdots, j)$, each composed of l_i $(k \geqq l_i \geqq 1)$ successive integers, so that

$$(15) \qquad \sum_{i=1}^{j} l_i = k.$$

Let the groups g_i be ordered in such a way that the integers contained in g_i are less than those contained in g_{i+1} $(i = 1, 2, \cdots, j-1)$. Thus the integers n_i immediately following the group of successive integers g_i satisfies

$$(16) \qquad n_i \geqq \sum_{r=1}^{i} l_r + 1$$

and

$$(17) \qquad 2^{-n_i-1}t \subset \mu_{l_i}.$$

Thus each group g_i of integers consisting of indices of large factors of the product (11) proves the existence of an integer n_i such that, by (13) and (17),

$$(18) \qquad n_i^{-1} \sin^2 2^{-n_i-1}t + \cos^2 2^{-n_i-1}t < n_i^{-1} + 2^{-2(l_i+1)}.$$

This inequality (18) is used to majorize j of the factors of the product (11) with indices among the $m-k$ values of n $(n = 1, 2, \cdots, m)$ for which $2^{-n-1}t$ is not in λ. The remaining $m-k-j$ $(\geqq 0)$ factors of this set are majorized by (12). Finally, all other factors are replaced by 1. Thus

$$| L(t;\tau)|^2 \leqq \delta^{m-k-j} \prod_{i=1}^{j} (n_i^{-1} + 2^{-2(l_i+1)})$$

or, using (16),

$$(19) \qquad | L(t;\tau)|^2 \leqq \delta^{m-k-j} \prod_{i=1}^{j} [(1 + \sum_{r=1}^{i} l_r)^{-1} + 2^{-2(1+l_i)}].$$

Now

$$(20) \quad [(1 + l_1)^{-1} + 2^{-2(1+l_1)}] [(1 + l_1 + l_2)^{-1} + 2^{-2(1+l_2)}] < \tfrac{9}{16}(1 + l_1 + l_2)^{-1} + 2^{-2(1+l_2)}(1 + l_1)^{-1} + 2^{-2(1+l_1+l_2)}.$$

Also, clearly,

$$(21) \qquad 2^{-2(1+l_2)}(1 + l_1)^{-1} < \tfrac{1}{4}(1 + l_1 + l_2)^{-1} < \tfrac{7}{16}(1 + l_1 + l_2)^{-1},$$

so that, by (20), (21),

$$(22) \qquad [(1 + l_1)^{-1} + 2^{-2(1+l_1)}] [(1 + l_1 + l_2)^{-1} + 2^{-2(1+l_2)}] < (1 + l_1 + l_2)^{-1} + 2^{-2(1+l_1+l_2)}.$$

By repeated use of the appraisal (22), formula (19) becomes

$$| L(t;\tau)|^2 < \delta^{m-k-j}[(1 + \sum_{i=1}^{j} l_i)^{-1} + 2^{-2(1+\sum_{i=1}^{j} l_i)}],$$

or

$$(23) \qquad | L(t;\tau)|^2 < \delta^{m-k-j} 2/k, \qquad m - k - j \geqq 0,\ j \leqq k.$$

If $k < m/2$, then, by (23),

$$| L(t;\tau)|^2 < [4(m\delta^m)/(2k\delta^{2k})](1/m) = O(1/m).$$

On the other hand, if $k \geqq m/2$,

$$| L(t;\tau)|^2 < \delta^0 \cdot 2 \cdot 2/m = O(1/m).$$

Thus, in either case, (7) follows from the definition (14) of the integer m.

To prove (8) let $t = 2^m \pi$. Then, by (11),

$$| L(2^m\pi;\tau)|^2 = m^{-1} \prod_{n=m+1}^{\infty} [n^{-1} \sin^2 2^{-n+m-1}\pi + \cos^2 2^{-n+m-1}\pi],$$

so that, neglecting the first term in each factor

$$| L(2^m\pi;\tau)|^2 > Cm^{-1} \qquad [C = \prod_{n=1}^{\infty} \cos^2(2^{-n}\pi/2) > 0].$$

This completes the proof of the italicized statement above.

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LIOUVILLE SYSTEMS AND ALMOST PERIODIC FUNCTIONS.*

By Aurel Wintner.

Since the Staude-Stäckel theory of conditionally periodic systems has been made standard through Charlier's textbook on celestial mechanics (1902; also 1927), and since Charlier's presentation follows closely that of Stäckel's paper of 1891, it is usually overlooked that, as pointed out by Stäckel himself (1905) and by Hadamard (1911), this simple and general theory leads to several difficulties. These difficulties of the standard theory are quite serious, since the objections of Stäckel and Hadamard, until they are met, prohibit any application of the Staude-Stäckel theory. In fact, the difficulty observed by Stäckel [7] is this: while the standard theory assumes that a certain Jacobian is distinct from zero in the whole domain under consideration, it turns out that this condition is necessarily violated even in the simplest possible cases (e. g., in the case of geodesics on a surface of Liouville type). The difficulty pointed out by Hadamard [5], being not of a local nature, is still more fundamental and concerns the usual introduction of the uniformizing variables into the (real) inversion problem of the Abel-Jacobi type; the possibility of defining these uniformizing variables, i. e., the problem of the monodromy group, being considered as settled by the local non-vanishing of the Jacobian involved.

In order to legitimize the procedure of the standard theory, Hadamard [5] has indicated direct topological considerations which, by proceeding from case to case, enable one to verify that the difficulties mentioned above can be disposed of. Furthermore, Stäckel [7] has shown by a detailed analysis of the uniformization belonging to bounded geodesics on surfaces of the Liouville type that in this particular case the local vanishing of the Jacobian, which is a necessity, cannot influence the correctness of the final result. Needless to say, such direct discussions can become quite involved, if one wants to take care of all the possible cases.

It will be shown in what follows that a straightforward and rigorous treatment of an extended class of separable systems can be based on Bohr's theory of almost periodic functions in such a way that no explicit discussions are needed when the theory is applied to concrete cases. While the class of systems to be considered does not include the most general separable system, it contains practically all of the classical integrable problems in the dynamics

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of particles, since it consists of the systems usually associated with the name Liouville. (Actually, there is no restriction at all in the fundamental case of $n = 2$ degrees of freedom, since it is known that in this case the most general separable system is a Liouville system, if one allows a coördinate transformation of the trivial type

$$\bar{x} = F_1(x) + F_2(y), \qquad \bar{y} = G_1(x) + G_2(y)$$

and assumes, as usual, that the system is reversible; cf. Stäckel [7], where further references are given.) That the problem of uniformization can be treated directly by means of the theory of almost periodic functions, is due to the fact that, in the case of Liouville systems, the inversion problem of the Abel-Jacobi type reduces to that of a synchronization of n given uniformizing parameters along n one-dimensional closed manifolds whose product space is the n-dimensional torus of the separable system.

Denoting by dots differentiations with respect to t, the general conservative Lagrangian function of the non-relativistic dynamical type and of n degrees of freedom is

$$(1) \qquad L \equiv L(x, \dot{x}) = \tfrac{1}{2} \Sigma \Sigma\, g_{ik} \dot{x}_i \dot{x}_k + \Sigma f_i \dot{x}_i + e, \qquad (\Sigma = \sum_1^n),$$

where $g_{ik} = g_{ki}, f_i, e$ are functions of $x = (x_1, \cdots, x_n)$ which have continuous partial derivatives of the second order in the x-domain under consideration, while the matrix $\| g_{ik} \|$ is positive definite in this domain. The reversible case is characterized by the identical vanishing of the n functions $f_i(x)$. Whether this condition is or is not satisfied, the Lagrangian equations have the energy integral

$$(2) \qquad \tfrac{1}{2} \Sigma \Sigma\, g_{ik}(x) \dot{x}_i \dot{x}_k - e(x) = h \qquad (h = \text{const.}).$$

In order to extend Liouville's type to the irreversible case, suppose that the $\frac{1}{2}n(n+1) + n + 1$ functions $g_{ik} = g_{ki}, f_i, e$ of the n variables $x_1, \cdots, x_n$ can be expressed in terms of n sets of 4 functions

$$(3) \qquad g_i = g_i(x_i), \quad f_i = f_i(x_i), \quad e_i = e_i(x_i), \quad d_i = d_i(x_i)$$

of the single variable x_i, where $i = 1, \cdots, n$, in such a way that the Lagrangian function (1) becomes †

† The content of the assumption (4) becomes clearer if one replaces the velocities $\dot{x}_i$ by the momenta $y_i = \partial L / \partial \dot{x}_i$ and L by the Hamiltonian function H. In fact, it is easily verified that (1) has the particular structure (4), (5) if and only if H is representable in terms of n pairs H_i^1, H_i^2 of functions of the two variables x_i, y_i in the symmetrical form

$$H(x_1, \cdots, x_n, y_1, \cdots, y_n) = \Sigma H_i^1(x_i, y_i) : \Sigma H_i^2(x_i, y_i).$$

$$(4)\qquad L \equiv L(x, \dot{x})$$
$$= \tfrac{1}{2} r(x_1, \cdots, x_n) \Sigma g_i(x_i) \dot{x}_i^2 + \Sigma f_i(x_i) \dot{x}_i + \Sigma e_i(x_i)/r(x_1, \cdots, x_n),$$

where

$$(5)\qquad r = r(x_1, \cdots, x_n) = \Sigma d_i(x_i).$$

Then the positive definite matrix $\| g_{ik} \|$ is the diagonal matrix formed by the products $r(x_1, \cdots, x_n) g_i(x_i)$, and so

$$(6_1)\quad r(x_1, \cdots, x_n) > 0; \qquad (6_2)\quad g_i(x_i) > 0, \qquad (i = 1, \cdots, n),$$

if the notations are chosen in a suitable way.

According to (6_1), one can introduce along any given solution path $x = x(t)$ a new time variable, t^*, by placing

$$(7)\quad t^* \equiv t^*(t) = \int^t d\bar{t}/r\big(x_1(\bar{t}), \cdots, x_n(\bar{t})\big), \text{ i. e., } \dot{t}^* = 1/r,\ t' = r,$$

where the prime denotes differentiation with respect to t^*. If the energy constant of $x = x(t)$ is h, put

$$(8)\qquad L^* = L^*(x, x'; h) = \tfrac{1}{2} \Sigma g_i x'^2_i + \Sigma f_i x'_i + \Sigma e_i + h \Sigma d_i.$$

The energy integral of the Lagrangian equations belonging to (8) is

$$(9)\qquad \tfrac{1}{2} \Sigma g_i(x_i) x'^2_i - \{\Sigma e_i(x_i) + h \Sigma d_i(x_i)\} = h^* \qquad (h^* = \text{const.}).$$

Now, those solutions $x = x(t)$ of the Lagrangian equations belonging to (4) which have the energy h are, in virtue of the time transformation (7) or its inverse

$$(10)\qquad t \equiv t(t^*) = \int^{t^*} r\big(x_1(\bar{t}^*), \cdots, x_n(\bar{t}^*)\big) d\bar{t}^*,$$

identical with those solutions $x = x(t^*)$ of the Lagrangian equations belonging to (8) which have the energy

$$(11)\qquad h^* = 0.$$

In order to see this, it is sufficient to observe that the Lagrangian function of the Maupertuis principle belonging to (1) is

$$\Lambda(x, \dot{x}; h) = [(e + h)(\Sigma\Sigma g_{ik} \dot{x}_i \dot{x}_k)]^{\frac{1}{2}} + \Sigma f_i x_i,$$

where, in view of (4), (5) and (6_1), (6_2),

$$[\ \ ]^{\frac{1}{2}} = [(r^{-1} \Sigma e_i + h)(r \Sigma g_i \dot{x}_i^2)]^{\frac{1}{2}} = [(\Sigma e_i + h \Sigma d_i)(\Sigma g_i \dot{x}_i^2)]^{\frac{1}{2}};$$

while $\Lambda(x, q\dot{x}; h) = q\Lambda(x, \dot{x}; h)$ for every $q > 0$, and so, in particular, for $q = r$ or $q = r^{-1}$.

The Lagrangian function (4) seems to be more general than a usual Lagrangian function of the Liouville type, since in (4) the terms linear in the velocities are not missing. Actually, one can omit the term $\Sigma f_i(x_i)\dot{x}_i$ without changing the Lagrangian equations, since $\Sigma f_i(x_i)dx_i$ is a complete differential. Thus, in contrast with the integrable irreversible dynamical systems investigated for $n=2$ by Birkhoff ([2], pp. 206-210), one can put $f_i(x_i)\equiv 0$ without loss of generality. Then the function (8) can be written as ΣL_i, where

$$(12_1)\quad L_i = \tfrac{1}{2}g_i(x_i)x'^2_i + U_i(x_i;h); \qquad (12_2)\quad U_i = e_i(x_i) + hd_i(x_i).$$

Hence the system of n Lagrangian equations $[L^*]_{x_i}=0$ splits into the n Lagrangian equations $[L_i]_{x_i}=0$ each of which has, in view of (12_1), a single degree of freedom, and, correspondingly, an energy integral

$$(13)\qquad \tfrac{1}{2}g_i(x_i)x'^2_i - U_i(x_i;h) = h_i \qquad (h_i = \text{const.}).$$

However, the choice of the n integration constants h_i is restricted by the condition

$$(14)\qquad \Sigma h_i = 0,$$

this condition being identical with (9), since $\Sigma h_i = h^*$ in view of (13), (12_2) and (9).

According to (6_2), one can write (13) in the form

$$(15_1)\quad x'^2_i = F_i(x_i;h_i,h); \qquad (15_2)\quad F_i = \frac{U_i(x_i;h)+h_i}{\tfrac{1}{2}g_i(x_i)}.$$

And $x_i = x_i(t^*)$ follows from (15_1), where $x'_i = dx_i/dt^*$, by the inversion of a quadrature. Since $x_i(t^*-t_i)$ is, for every solution $x_i(t^*)$ of $[L_i]_{x_i}=0$ and for every constant t_i, a solution of $[L_i]_{x_i}=0$, the integration constant introduced by the quadrature can be omitted, so that the motion is, in the main, determined for every i by the energy constant h_i alone. In this sense, it is permissible to denote the general solution by

$$(16)\qquad x_i = x_i(t^*;h_i,h),$$

where one can add to t^* different arbitrary constants for different values of i.

Suppose, for simplicity, that the x_i-space is the whole space $-\infty < x_i < +\infty$, where $i=1,\cdots,n$, and assume that if $F_i(x_i;h_i,h)>0$ for large $x_i>0$ or large $-x_i>0$, then

$$\int^{\pm\infty} F_i(\bar{x}_i;h_i,h)^{-\frac{1}{2}}\,d\bar{x}_i = \pm\,\infty \quad (\text{e.g., } |F_i| < \text{const.}\, x_i^2).$$

Since, from (15_1),

$$(17) \qquad t^* = \int^{x_i} \{F_i(\bar{x}_i; h_i, h)\}^{-\frac{1}{2}} d\bar{x}_i,$$

the greatest lower bound, α_i, and the least upper bound, β_i, of (16) for $-\infty < t^* < +\infty$ are two *subsequent* roots

$$(18) \qquad x_i = \alpha_i \equiv \alpha_i(h_i, h), \qquad x_i = \beta_i \equiv \beta_i(h_i, h)$$

of the equation

$$(19) \qquad F_i(x_i; h_i, h) = 0$$

of Hill's manifold of zero velocity [cf. (15_1)], where it is understood that $\alpha_i = -\infty$ and/or $\beta_i = +\infty$ in case such roots do not exist, and that $\alpha_i = \beta_i$ if and only if (16) is independent of t^*. If (16) is for some t^* between two subsequent distinct multiple roots (18) of the equation (19), then the integrand of (17) becomes infinite in a non-integrable order ($\geqq 1$) when $x_i \to \alpha_i + 0$ (or $x_i \to \beta_i - 0$), and so (16) tends to α_i (or β_i) either when $t^* \to -\infty$ or when $t^* \to +\infty$ or when $t^* \to \pm\infty$. These asymptotic cases will be excluded in what follows. Then there exist two subsequent simple roots (18) of (19) such that α_i is the minimum and β_i the maximum of (16) for $-\infty < t^* < +\infty$, the integrand of (17) becoming infinite at $x_i = \alpha_i$ and $x_i = \beta_i$ in the integrable order $\frac{1}{2}$. Thus, from (18) and (19),

$$(20) \qquad F_i(x_i; h_i, h) = (\beta_i - x_i)(x_i - \alpha_i)G_i, \qquad (\alpha_i < \beta_i),$$

where, by (15_1),

$$(21) \qquad G_i(x_i; h_i, h) > 0 \quad \text{for} \quad \alpha_i \leqq x_i \leqq \beta_i,$$

and G_i remains continuous at $x_i = \alpha_i, \beta_i$. Let h and h_i be fixed for a fixed i, so that (16), (20), (21) reduce to $x_i(t^*)$, $F_i(x_i)$, $G_i(x_i)$ respectively.

It is easily inferred from (17), (20), (21) that (16) is periodic with the primitive period

$$(22) \qquad \tau_i = 2\int_{\alpha_i}^{\beta_i} \{F_i(\bar{x}_i)\}^{-\frac{1}{2}} d\bar{x}_i.$$

One can uniformize the relation (17) between t^* and x_i in the usual manner (Abel, Hill, Weierstrass) in terms of a uniformizing time variable u_i which varies with t^* from $-\infty$ to $+\infty$ and reduces to the eccentric anomaly in case of Kepler's motion (where $n = 1$, $x_i = x_1$ = radius vector, t^* = time and the degree of freedom is reduced to $n = 1$ by considering the true anomaly as a coördinate which is ignorable in the sense of Routh). This uniformization of (17) is

$$(23_1) \quad x_i = \tfrac{1}{2}(\beta_i + \alpha_i) - \tfrac{1}{2}(\beta_i - \alpha_i)\cos u_i; \qquad (23_2) \quad t^* = \tfrac{1}{2\pi}\tau_i u_i + P_i(u_i),$$

where τ_i is the fixed period (22) and the function $P_i(u)$ is such that

$$(24_1) \quad dt^*/du_i > 0; \qquad (24_2) \quad P_i(u_i + 2\pi) = P_i(u);$$

(it is essential that in (24_1) the sign of equality, which would be compatible with a strictly increasing function (23_2) of u_i, is excluded for every u_i). For let the derivative (24_1), considered as a function of u_i, be defined as the positive square root of the function which one obtains by substituting the function (23_1) of u_i into the positive continuous function (21) of x_i. Then the function (24_1) of u_i has the period 2π and over this period a positive mean value, say γ_i. Hence, $t^* = \gamma_i u_i + P_i(u_i)$ holds for a function $P_i(u_i)$ which satisfies (24_2). In order to see that this implies the uniformization of the relation (17) in the form (23_1), (23_2), where $-\infty < u_i < +\infty$, it is sufficient to compare (20) with the representation (22) of the period of (16), and to observe that (23_1) is equivalent to $(dx_i/du_i)^2 = (\beta_i - x_i)(x_i - \alpha_i)$, while $\min x_i(t^*) = \alpha_i$, $\max x_i(t^*) = \beta_i$.

The n periods τ_i with respect to t^* are, in view of (18), (20) and (22), continuous functions of the integration constants h_i, h and are, therefore, incommensurable in general. Hence the uniformization of the solutions $x_i = x_i(t)$ of the unseparated system of the original Lagrangian equations $[L]_{x_i} = 0$ is not an elementary task. For in order to obtain the n functions $x_i = x_i(t)$, where $-\infty < t < +\infty$, one has to eliminate the $n+1$ intermediary time variables u_i, t^* between the $2n+1$ equations (23_1), (23_2), (10). This non-local elimination will be carried out by using a theorem on almost periodic functions, almost periodicity being meant in the original sense of Bohr.

If $\phi = \phi(t^*)$, $-\infty < t^* < +\infty$, is a real almost periodic function which has a derivative $\phi'(t^*) > -1$ for every t^*, then, $\phi(t^*)$ being bounded, the function $t = t^* + \phi(t^*)$ of t^* varies with t^* from $-\infty$ to $+\infty$ in a strictly increasing manner. This does not imply that the almost periodic function $t - t^*$ of t^* is an almost periodic function of t. If, however, the assumption $-1 < \phi'(t^*)$ is replaced by the sharper assumption that there exists a ϑ for which

$$(25) \quad -1 < -\vartheta \leqq \phi'(t^*), \text{ where } \vartheta = \text{const. for } -\infty < t^* < +\infty,$$

then, on placing

$$(26) \qquad t^* = t + \psi(t), \text{ where } t = t^* + \phi(t^*),$$

the almost periodicity of $\phi(t^*)$ implies the almost periodicity of $\psi(t)$; furthermore, the moduls determined by the frequencies of $\phi(t^*)$ and $\psi(t)$ are identical (cf. Bohr [3], where it is assumed that $|\phi'(t^*)| \leqq \vartheta < 1$, while actually (25) is sufficient; cf. Bohr and Jessen [4]).

Returning to the uniformization (23_1), (23_2), where τ_i is a positive constant and the correspondence between $-\infty < u_i < +\infty$ and $-\infty < t^* < +\infty$ is topological, it is clear that the inverse of the function (23_2) can be written in the form

$$(27_1)\quad u_i = \sigma_i t^* + Q_i(t^*); \qquad (27_2)\quad \sigma_i = 2\pi/\tau_i,$$

where, in view of (24_1) and (24_2),

$$(28_1)\quad u'_i > 0, \text{ i. e., } Q'_i(t^*) > -\sigma_i; \qquad (28_2)\quad Q_i(t^* + \tau_i) = Q_i(t^*).$$

Since $x_i(t^* + \tau_i) = x_i(t^*)$, and since the functions (3) depend only on x_i, one has

$$(29)\qquad d_i(t^* + \tau_i) = d_i(t^*), \text{ where } d_i(t^*) = d_i(x_i(t^*)).$$

Hence (5) is a sum of n functions of t^* which have the periods $\tau_1, \cdots, \tau_n$ respectively, so that

$$(30)\qquad r(t^*) = \Sigma d_i(t^*)$$

is almost periodic. From (10) and (30) one has

$$(31_1)\quad t \equiv t(t^*) = \int^{t^*} r(\bar{t}^*)d\bar{t}^* = \Sigma s_i(t^*); \qquad (31_2)\quad s_i(t^*) = \int^{t^*} d_i(\bar{t}^*)d\bar{t}^*.$$

Let, as usual, $M\{f\}$ denote the mean value of an almost periodic (e. g., continuous periodic) function f, i. e., the constant term in the Fourier expansion of f. It is clear from (31_2) and (29) that the difference of $s_i(t^*)$ and $t^*M\{d_i\}$ has the period τ_i. It follows, therefore, from (31_1), (30) that, if $\mu = \Sigma M\{d_i\}$, one has

$$(32_1)\quad t = \mu t^* + v(t^*); \qquad (32_2)\quad v(t^*) = \Sigma p_i(t^*),$$

where

$$(33_1)\quad p_i(t^* + \tau_i) = p_i(t^*) \equiv s_i(t^*) - t^*M\{s_i\}; \qquad (33_2)\quad \mu = M\{r\}.$$

Since (5) is, by (6_1), everywhere positive, and since (5) is a continuous function of the position in the space $x = (x_1, \cdots, x_n)$, it has on every bounded subset of this space a non-vanishing greatest lower bound. On the other hand, (29) shows that (30) is obtained by substituting into the function (5) of $(x_1, \cdots, x_n)$ the functions $x_i = x_i(t^*)$, functions which are continuous and periodic, hence bounded, for $-\infty < t^* < +\infty$. Consequently, if m denotes the greatest lower bound of the almost periodic function $r(t^*)$ for $-\infty < t^* < +\infty$, then $m \neq 0$, i. e., $m > 0$. Since $m \leqq \mu$ in view of (33_2), it follows that $\mu > 0$. Accordingly, one can assume without loss of generality that $\mu = 1$

(correspondingly, one could have modified the definition (7) of t^* by writing cr instead of r, where c is an arbitrarily positive constant). Thus

$$(34) \qquad 0 < m = \text{fin inf}\, r(t^*) \leqq M\{r\} = \mu = 1.$$

Let $\phi(t^*)$ denote the function (32_2) in case of the normalization $\mu = 1$, so that, from (32_1),

$$(35) \qquad t \equiv t(t^*) = t^* + \phi(t^*).$$

According to (33_1), (32_2), the function $\phi(t^*) \equiv v(t^*)$ is almost periodic and has frequencies contained in the modul of the n (not necessarily linearly independent) numbers σ_i which are defined by (27_2).

It follows that one can define for $-\infty < t < +\infty$ a unique function $\psi(t)$ by the requirement (26), and that this $\psi(t)$ is almost periodic and has frequencies which are contained in the modul generated by the σ_i. In fact, $t'(t^*) = r(t^*)$ by (10), while $t'(t^*) = 1 + \phi'(t^*)$ by (35). Hence (34) can be written in the form

$$(36) \qquad 0 < m = 1 + \text{fin inf}\, \phi'(t^*) \leqq 1.$$

Now there are two cases possible according as $m \neq 1$ or $m = 1$. If $m \neq 1$, then, from (36),

$$m - 1 = \text{fin inf}\, \phi'(t^*), \text{ where } -1 < m - 1 < 0.$$

This means that (25) is satisfied by $\vartheta = 1 - m$, and so one can apply the general theorem concerning the inversion (26) of (35). If, on the other hand, $m = 1$, then (34) reduces to

$$1 = \text{fin inf}\, r(t^*) = M\{r\},$$

so that $r(t^*) \equiv 1$ by the uniqueness theorem of almost periodic functions. Hence it is seen from (10) that the exceptional case $m = 1$ belongs to the trivial case where $\psi(t)$ in (26) is a constant.

Since, from (27_1) and (26),

$$u_i = \sigma_i t + \sigma_i \psi(t) + Q_i(t + \psi(t)), \text{ where } Q_i(t + \psi(t) + \tau_i) = Q_i(t + \psi(t)),$$

and since the frequencies of the almost periodic function $\psi(t)$ are contained in the modul generated by the $\sigma_i = 2\pi/\tau_i$, where $i = 1, \cdots, n$, it is clear from (23_1) that each of the n components $x_i(t)$ of the solution of the original Lagrangian system $[L]_{x_i} = 0$ is an almost periodic function whose frequencies

are contained in the modul of the σ_i. In other words, the $x_i(t)$ possess an anharmonic Fourier analysis of the form

$$x_i(t) = \Theta_i(t/\tau_1, \cdots, t/\tau_n), \qquad (i = 1, \cdots, n), \tag{37}$$

where the

$$\Theta_i = \Theta_i(\theta_1, \cdots, \theta_n), \qquad (i = 1, \cdots, n), \tag{38}$$

are continuous functions of the position on the n-dimensional torus

$$0 \leqq \theta_1 < 1, \cdots, 0 \leqq \theta_n < 1,$$

while the $\tau_i = 2\pi/\sigma_i$ are the positive constants defined by (22). This is the desired result.

It is, in this connection, natural to ask whether or not the continuous function $\Theta = \Theta(\theta_1, \cdots, \theta_n)$ on the torus is necessarily regular analytic whenever it belongs to an almost periodic function $x(t)$ which is regular analytic and bounded in a strip about the t-axis and has n linearly independent frequencies. This problem seems to be quite difficult, since it is, in the main, a generalization of the (unsolved) problem of Poincaré-Denjoy in the analytic case ($n = 2$). The solution of the problem would be essential also for Levi-Civita's recent theory [6] of conditionally periodic systems, since, in general, nothing is known about $\Theta(\theta_1, \cdots, \theta_n)$ except for its continuity, so that not even Birkhoff's formal approach [1] to the Weierstrass preparation theorem is applicable.

As mentioned in the introduction, Stäckel [7] has found that a certain Jacobian cannot be distinct from zero for every t. A comparison of the procedure of the present paper with the calculations of Stäckel shows that the vanishing of the Jacobian in question is obvious without any particular calculations, since it is nothing but a manifestation of Hill's manifold of zero velocity. Correspondingly, the calculations and verifications of Stäckel [7] can be avoided by realizing that Hill's manifold of zero velocity cannot be reached by a path in a direction distinct from the transversal direction.†

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† This fact, well known in case of the restricted problem of three bodies (a case in which the transversal is the normal, the metric being Euclidean), holds in the general case (1) and can be proved as follows: Let the initial conditions x^0, x'^0 assigned to $t = 0$ be such that the vector x'^0 vanishes, so that x^0 is, by (2), a point of Hill's manifold $e(x) + h = 0$ of zero velocity belonging to $h = -e(x^0)$. For typographical reasons, differentiations with respect to t are denoted by primes instead of by dots. In order to exclude the case of an equilibrium solution $x(t) = \text{const.}$ represented by the

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single point $x = x^0$, one has to assume that grad $e(x)$ does not vanish at $x = x^0$. Then the hyper-surface $e(x) + h = 0$ has at $x = x^0$ an orientable normal and

$$\Sigma\Sigma\, g^{ik}(x^0)\, e_{x_i}(x^0)\, e_{x_k}(x^0) > 0.$$

Transversality being meant with reference to the Riemannian geometry of the $g_{ik}(x)$ which determine the quadratic part of (1), one has to prove that

$$\frac{|\,\Sigma\, x'_i(t)\, e_{x_i}(x^0)\,|}{\{\Sigma\Sigma\, g_{ik}(x^0)\, x'_i(t)\, x'_k(t)\}^{\frac{1}{2}}\,\{\Sigma\Sigma\, g^{ik}(x^0)\, e_{x_i}(x^0)\, e_{x_k}(x^0)\}^{\frac{1}{2}}} \to 1 \text{ as } t \to \pm 0,$$

where $x(t)$ is the solution for which $x(0) = x^0$, $x'(0) = x'^0 =$ zero vector. Since $\|\, g^{ik} \,\| = \|\, g_{ik} \,\|^{-1}$, it follows that it is sufficient to prove the relation

$$x'_i(t) = t \sum_k g^{ik}(x^0)\, e_{x_k}(x^0) + o(|\,t\,|), \text{ where } t \to \pm 0.$$

Now $x'_i(t) = x''_i(0)t + o(|\,t\,|)$, by Taylor's theorem. Hence it is sufficient to prove that

$$x''_i(0) = \sum_k g^{ik}(x^0)\, e_{x_k}(x^0), \text{ i.e., } e_{x_i}(x^0) = \sum_k g_{ik}(x^0)\, x''_k(0), \text{ where } i = 1, \cdots, n.$$

And the truth of the last relation follows by placing $t = 0$ (hence $x'^{\,0}_i = 0$) in the Lagrangian equations belonging to (1). In fact, these equations are easily found to be

$$\sum_k g_{ik} x''_k + \sum_j \sum_k \Gamma_i{}^{jk} x'_j x'_k + \sum_k \Pi_{ik} x'_k - e_{x_i} = 0, \qquad (i = 1, \cdots, n),$$

where the $\Gamma_i{}^{jk} = \Gamma_i{}^{kj}$ are the Christoffel symbols of the first kind belonging to the g_{ik}, while the $\Pi_{ik} = -\Pi_{ki}$ are the components of the alternating derivative (curl) of the covariant vector $(f_1, \cdots, f_n)$.

GALILEI GROUP AND LAW OF GRAVITATION.*

By Aurel Wintner.

It is known that among all attraction laws in which the force is proportional to a fixed, say the β-th, power of the distance, the case $\beta = -3$ is exceptional in many respects. For instance, in the case of two bodies all non-circular solutions lead, if $\beta = -3$, on the one hand to a collision and on the other hand to a recession to infinity. In the case of three bodies, Lagrange [1] has shown that in Newton's case $\beta = -2$ all homographic solutions are coplanar solutions; and this holds, according to Banachiewitz,[2] for every $\beta \neq -3$ but not for $\beta = -3$. The object of the following considerations is a general elucidation of the exceptional behavior of the inverse cubic law of attraction.

Since the ten classical integrals do not depend on the choice of the law of attraction and are due, according to Jacobi and Lie, to the infinitesimal trans-

* Received January 17, 1938.

[1] This is the main theorem of Lagrange concerning the homographic solutions of the problem of three bodies; cf., the concluding remarks on p. 292 of vol. 6 (1873) of Lagrange's collected works. A simple proof of the theorem is due to Pizzetti ("Casi particolari del problema dei tre corpi," *Rendiconti della Reale Accademia dei Lincei*, ser. 5, vol. 13_1 (1904), pp. 17-26), who also proved the more general theorem that every homographic solution of the problem of $n \geqq 2$ bodies is either coplanar or homothetic. This method of Pizzetti was rediscovered by Müntz (*Mathematische Zeitschrift*, vol. 15 (1922), pp. 169-187) and recently by Carathéodory (*Sitzungsberichte der Bayerischen Akademie der Wissenschaften*, 1933, pp. 257-267).

[2] T. Banachiewitz, "Sur un cas particulier du problème des trois corps," *Comptes Rendus*, vol. 142 (1906), pp. 510-512. The discussion of geometrical compatibility is not carried out by Banachiewitz but becomes quite easy by using the important fact, apparently not observed, that the non-planar solutions of Banachiewitz are isosceles (but not equilateral) solutions. For let $(a_i, b_i, 0)$ be the initial values of the barycentric coördinates (x_i, y_i, z_i) of the mass m_i, where $i = 1, 2, 3$, and let r_i denote the initial length $\{(a_j - a_k)^2 + (b_j - b_k)^2\}^{\frac{1}{2}}$ of a side of the triangle formed by the three bodies. Banachiewitz assumes that the three pairs (a_i, b_i) satisfy the conditions

$$\Sigma\, m_i a_i b_i = 0, \quad \Sigma\, m_i a_i b_i / r_i{}^4 = 0; \qquad \Sigma\, m_i a_i = 0, \quad \Sigma\, m_i b_i = 0,$$

and that the three points (a_i, b_i) of the initial plane are neither collinear nor such that $r_1 = r_2 = r_3$. Now, it is easily shown either by a determinant calculation or by an equivalent geometrical consideration that these conditions cannot be satisfied unless at least one of the $3 + 3$ numbers a_i, b_i vanishes, in which case direct substitution of $a_3 = 0$ into the above conditions shows that $r_1 = r_2$ (and $b_1 = b_2$, $m_1 = m_2$). Since the solution is homographic, it follows that the triangle is isosceles not only at $t = 0$ but for every t.

formations which generate the Galilei group,[3] it is clear that in the case of suitably chosen laws of attraction the group of transformations of the equations of motion is larger than the Galilei group. (If, for instance, $\beta = 2$, the problem of n bodies can be split into n vectorial harmonic oscillators and admits, therefore, certain continuous groups of orthogonal substitutions which it does not admit in case of another attraction law). Of course, this cannot be seen from the usual introduction of the Galilei group, since usually it is merely stated, but not proved, that the Galilei group exhausts the automorphic transformations of the Newtonian problem of n bodies. It will be seen that this implication concerning the completeness of the Galilei group, while (in the main) correct, is by no means so evident as to need no proof. In fact, it turns out that if the attraction were proportional to the —3rd, instead of the —2nd, power of the distance, the Galilei group would be only a subgroup of all "inertial" transformations involving the time.

Without assuming the existence of any of the ten classical integrals, consider a system of $k(=3n)$ differential equations of the second order, $x_i'' = f_i$, where $' = d/dt$ and $i = 1, \cdots, k$, and suppose that the k given continuous functions f_i are independent of t and are homogeneous, of some fixed degree β, in $(x_1, \cdots, x_k)$. For instance, $\beta = -2, 1, -1$ in the cases of Newton, Hooke and a logarithmical potential respectively. It is natural to seek pairs of *fixed* functions $u = u(t)$, $v = v(t)$ which have the property that $x_i = v(t)x_i(u(t))$ is, for *every* solution $x_i = x_i(t)$ of $x_i'' = f_i$, again a solution ("dynamical similarity"). It will be assumed that the solutions of $x_i'' = f_i$ are uniquely determined by the initial values and that $u(t)$, $v(t)$ have continuous second derivatives u'', v'', finally that v and the first derivative of u are positive on the t-interval under consideration. In particular, one can introduce $u = u(t)$ instead of t as an independent variable, so that $t = t(u)$.

Since the $f_i(x_1, \cdots, x_k)$ are homogeneous of degree β, it is found by direct substitution that if $x_i = x_i(t)$ is a fixed solution of $x_i'' = f_i$, where $x_i'' = d^2x_i/dt^2$, then $x_i = v(t)x_i(u(t))$ is again a solution if and only if

$$v^\beta \frac{d^2x_i}{du^2}\left(\frac{dt}{du}\right)^3 = \frac{d^2(vx_i)}{du^2}\,\frac{dt}{du} - \frac{d(vx_i)}{du}\,\frac{d^2t}{du^2}$$

is, in virtue of $t = t(u)$, an identity in u. It follows, therefore, by comparison of the coefficients of d^jx_i/du^j, where $j = 0, 1, 2$, that the two functions u, v

[3] Cf., F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, vol. II (1927), pp. 53-59.

of t will have the desired property with reference to every solution $x_i = x_i(t)$ of $x_i'' = f_i$ if the two functions t, v of u satisfy the three conditions

$$v/\dot{t}^2 = v^\beta, \qquad 2\dot{v}\dot{t} - v\ddot{t} = 0, \qquad \ddot{v}\dot{t} - \ddot{t}\dot{v} = 0,$$

where the dots denote differentiations with respect to u. The third of these conditions means that $\dot{v}/\dot{t}$ is a constant, say c. Hence, on differentiating the first condition with respect to u and substituting the representation of $\ddot{t}$, thus obtained, into the second condition, it is seen that the three conditions can be written as follows:

$$\text{(I)}\quad \dot{t}^2 = v^{1-\beta}; \qquad \text{(II)}\quad 4\dot{t}^2\dot{v} = (1-\beta)v^{1-\beta}\dot{v}; \qquad \text{(III)}\quad \dot{v} = c\dot{t}.$$

Since $v > 0$ and $\dot{t} > 0$ by assumption, (I) implies that (II) is satisfied if and only if either $\beta = -3$ or $\dot{v} \equiv 0$.

Suppose first that $\dot{v} \equiv 0$ (for which $\beta \neq -3$ is sufficient but not necessary). Then (II) is an identity, while (III) is satisfied by $c = 0$ (and, since $\dot{t} > 0$, only by $c = 0$); so that the three conditions reduce to (I). Now $\dot{v} \equiv 0$, i. e., v is a (positive) constant, say λ. Hence, the single condition (I), where $\dot{t} = dt/du$, gives $u = \lambda^{\frac{1}{2}(\beta-1)}t$ plus an additive constant (which is unessential, since t does not occur explicitly in $x_i'' = f_i$). This determines the most general pair $u(t)$, $v(t)$ $(\equiv \lambda)$, if $\beta \neq -3$. Choosing, in particular, $\beta = -2$, it follows that $x_i = \lambda x_i(\lambda^{-3/2}t)$ is, for every positive constant λ and for every solution $x_i = x_i(t)$ of the problem of $n(=\frac{1}{3}k)$ bodies, a solution of the problem of n bodies.

This trivial transformation group, while not contained by the Galilei group, does not lead to an eleventh integral, a failure which, for reasons explained by Engel,[4] does not contradict Lie's integration theory.

Now let $\beta = -3$. Then (II) is, in virtue of (I), an identity also when $\dot{v} \not\equiv 0$, so that one can choose the constant c of (III) distinct from 0. Thus the three conditions (I), (II), (III) for the function pair $u = u(t)$, $v = v(t)$ reduce to the pair of conditions $u'^2 = v^{\beta-1}$, $v'\dot{t} = c\dot{t}$, or, since $\beta = -3$ and $\dot{t} > 0$, to $u' = v^{-2}$, $v' = c$; so that

$$u = u(t) = \int_0^t (c\bar{t} + b)^{-2} d\bar{t} + a, \qquad v = v(t) = ct + b,$$

where $a, b, c(\neq 0)$ are arbitrary constants. In particular, $x_i = -tx_i(t^{-1})$ is,

[4] F. Engel, "Nochmals die allgemeinen Integrale der klassischen Mechanik," *Göttinger Nachrichten, 1917*, pp. 189-198.

for every solution $x_i = x_i(t)$ of $x_i'' = f_i$, again a solution,[5] and so the group of inertial transformations is essentially larger than the Galilei group.

This result seems to be of interest also because it explains *why* Jacobi was able to find, for the problem of n bodies in case of inverse cubic attraction, two new first integrals.[6] These allowed him to reduce the rectilinear problem belonging to $n = 3$ to a quadrature.[7]

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[5] This fact corresponds to the results of Cunningham and Bateman concerning the invariance of Maxwell's equations under certain transformations by reciprocal radii; cf. F. Klein, *loc. cit.*[3], pp. 78-79.

[6] Jacobi, *Gesammelte Werke*, Supplementband (1884), p. 27.

[7] Jacobi, *loc. cit.*[6], vol. 4 (1886), pp. 481-484 and pp. 533-539.

INTERIOR TRANSFORMATIONS ON SURFACES.*

By G. T. Whyburn.

A single valued continuous transformation $T(A) = B$ is said to be *interior*[1] provided the image of every open set in A is open in B; and such a transformation is said to be *light*[2] provided the inverse set of every point in B is totally disconnected.

Stoilow[1] has analyzed light interior transformations under the assumption that both A and B are regions on a sphere or plane. Thus in this important case we have an analysis of interior light transformations from one (open) 2-dimensional manifold to another. In the present paper the principal theorem to be established (see § 2) is to the effect that if A is a compact 2-dimensional manifold (with or without boundary curves), so also is any light interior image of A. Thus the 2-dimensional manifold character of a set appears as an invariant under light interior transformations and accordingly it need not be assumed for the image set.

Various applications of our principal theorem will follow in §§ 3 and 4. Notably, in case A is a sphere, it is shown that any light interior image of A is necessarily either a sphere, projective plane or 2-cell; and furthermore each true cyclic element of *any interior image* of A (whether the transformation is light or not) is a sphere, projective plane or 2-cell.

1. Preliminary lemmas and theorems. All sets considered are assumed to lie in a separable metric space. We begin with

(1.1). Lemma. *If M is a locally connected locally compact continuum having no local separating point, each point $p \in M$ is contained within an arc qpr of M which does not separate M.*

Proof. There exists[3] an uncountable aggregate X of arcs $[px]$ in M each pair of which intersect in just p. Since the aggregate of disjoint connected

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[1] See Stoilow, *Annales Scientifiques de l'Ecole Normale Supérieure*, vol. 63 (1928), pp. 347-382 and *Annales de l'Institut Henri Poincaré*, vol. 2 (1932), pp. 233-266.

[2] See my paper in the *Duke Mathematical Journal*, vol. 3 (1937), pp. 370-381.

[3] See G. T. Whyburn, *American Journal of Mathematics*, vol. 53 (1931), pp. 305-314; see also an abstract by Zippin in the *Bulletin of the American Mathematical Society*, vol. 36 (1930), p. 805.

subsets $[px - p]$ of the connected set $M - p$ is uncountable it therefore contains [4] an uncountable subaggregate Y such that for each set $py - p$, in Y, $(M - p) - (py - p)$ has at most two components and every such component is bounded by the entire set $py - p$. On any one such arc py choose an interior point q. Then clearly $M - pq = (M - p) - (pq - p)$ is connected. Let Z denote the collection obtained by omitting from Y the set containing q. Then since Z is an uncountable collection of disjoint connected subsets of the connected set $M - pq$ it therefore contains at least one element, say $pz - p$ such that $(M - pq) - (pz - p)$ has at most two components and each of these is bounded (rel. $M - pq$) by all of $pz - p$. Let r be any interior point of pz. Then clearly

$$(M - pq) - (pr - p) = M - (pq + pr) = M - qpr$$

is connected.

(1.2). *Let $T(A) = B$ be interior and light where A is connected and locally connected and is locally a cantorian manifold* [5] *of dimension ≥ 2. Then B is also locally a cantorian manifold of dimension ≥ 2.*

Proof. For suppose some connected open subset Q of B is separated by a compact 0-dimensional subset X'. Then X' contains a closed subset X which irreducibly separates Q between some two points a and b. Let $y \in X$, $x \in T^{-1}(y)$ and let V be a connected neighborhood of x so chosen that $T(V) \subset Q$ and $F(V) \cdot T^{-1}(X) = 0$.[6] (Note $T^{-1}(X)$ is of dimension 0 since T is continuous and light.) Let $S = V - V \cdot T^{-1}(X)$. Then S is connected and open and hence so also is its image. But $T(S) \subset Q - X$ and since $T(S)$ must intersect both the component of $Q - X$ containing a and the one containing b it follows that X does not separate a and b in Q. Thus we have a contradiction and our result is proven.

(1.21). Corollary. *Let $T(A) = B$ be continuous and light where A is locally connected and is locally a cantorian manifold of dimension ≥ 2. If T is interior at the point $x \in A$, then $T(x)$ is not a local separating point of B. Thus in particular, if T is interior on A, B has no local separating points.*

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[4] See my paper in the *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 444-454.

[5] A connected and locally connected set A is said to be locally a cantorian manifold of dimension ≥ 2 provided no connected open subset of A is separated by a compact, totally disconnected set.

[6] For any open set V, $F(V)$ denotes the set-theoretic boundary of V, i.e., the set $\bar{V} - V$. For any set X and any number $\delta > 0$, $V_\delta(X)$ denotes the set of all points at a distance $< \delta$ from X.

(1.3). THEOREM. *Let $T(A) = B$ be interior and light where A is a 2-dimensional manifold (with or without boundary curves). Then for each $b \in B$, $T^{-1}(b)$ is finite.*

Proof. By (1.21) B has no local separating point. Hence by (1.1) each point $b \in B$ is interior to some arc pbq in B such that $B - pbq$ is connected. Let $X = T^{-1}(pbq)$. Then $A - X$ can have only a finite number of components [2] and furthermore X is a locally connected set.[2] Hence X can contain at most a finite number of simple closed curves. Thus if C is any component of X, each true cyclic element of C is a graph and there are only a finite number of such cyclic elements. Accordingly $T^{-1}(b)$ can intersect each cyclic element of C in just a finite number of points and thus can contain only a finite number of points of C which are on true cyclic elements of C. But since b is of order 2 in pbq it follows that $C \cdot T^{-1}(b)$ contains no end points of C; and since $pbq - b$ has just two components, $T^{-1}(b) \cdot C$ can contain only a finite number of cut points of C. Therefore $T^{-1}(b) \cdot C$ is finite and hence so also is $T^{-1}(b) \cdot X = T^{-1}(b)$, since X has only a finite number of components C.

(1.4). *If A is a 2-cell with boundary J, if $T(A) = B$ is interior and light and if C is a simple closed curve in B such that $T^{-1}(C) = J$, then B is a 2-cell with boundary C.*

Proof. By virtue of Zippin's characterization of the 2-cell [7] we have to prove that every arc spanning C in B irreducibly separates C.

To that end let cvd be any such arc and let x and y be points separating c and d on C. We first show that cvd separates x and y in B. If this is not so we may suppose we have an arc $xy \subset B - cvd$ so that $xy \cdot J = x + y$. Let c_1 and d_1 be points of $T^{-1}(c)$ and $T^{-1}(d)$ respectively. An arc c_1d_1 of J contains at least one point of $T^{-1}(x)$ or $T^{-1}(y)$, say a point x_1 of $T^{-1}(x)$. There exists an arc x_1y_1 in $T^{-1}(xy)$ which maps topologically onto xy.[2] Since there exists similarly an arc d_1c_2 in $T^{-1}(cvd)$ with $c_2 \subset T^{-1}(c)$, it follows that the arc $x_1d_1y_1$ of J contains c_2. The subarc c_2d_1 of $x_1d_1y_1$ must contain a point of $T^{-1}(x)$ or $T^{-1}(y)$, say a point x_2 of $T^{-1}(x)$. Then, repeating the argument, it follows that c_2d_1 also contains a point y_2 of $T^{-1}(y)$; and then that the subarc x_2y_2 of c_2d_1 contains points c_3 and d_2 of $T^{-1}(c)$ and $T^{-1}(d)$ respectively and so on. Clearly this is impossible, since each of the sets $T^{-1}(c)$, $T^{-1}(d)$, $T^{-1}(x)$ and $T^{-1}(y)$ is finite. Hence cvd separates x and y in B. Clearly no subarc of cvd can separate x and y in B. Thus since B is unicoherent,[8] no proper

[7] *American Journal of Mathematics*, vol. 55 (1933), pp. 201-217.

[8] See Eilenberg, *Fundamenta Mathematicae*, vol. 24 (1935), p. 175

subset of cvd separates x and y in B; and since clearly there cannot exist three finite sets of components of $A - T^{-1}(cvd)$ each having a subcontinuum of $T^{-1}(cvd)$ on its boundary, it follows that cvd separates B into just two components and hence separates it irreducibly. Accordingly B is a 2-cell with boundary C.

We conclude this section with a lemma which, while not used in the proof of our main result, will be useful in some of the applications.

(1.5). LEMMA. *Let $T(A) = B$ be interior, let R be an open subset of A with boundary F, let $B_0 = T(\bar{R})$ and designate the transformation $T(\bar{R}) = B_0$ by T^*. Let $E \subset F$ be the set of points (if any) where T^* fails to be interior. Then for any $x \in E$ and any neighborhood U of x, $T(E \cdot U)$ locally separates B_0 at the point $T(x)$.*

Proof. Let $y = T(x)$, let $\epsilon > 0$, let U be any open subset of A containing x such that $\bar{U} \subset V_\epsilon(x)$ and let $U_0 = U \cdot \bar{R}$. Finally let $V = T(U_0)$. Now if we suppose $T(E \cdot U)$ does not locally separate B_0 at y, there will exist an open subset W of B_0 containing y and such that $W \subset T(U)$ and $W - W \cdot T(E) = X$ is connected. Then we must have $X \subset V$. For if not there will exist a point $z \in X \cdot V(\overline{X - X \cdot V}) + \overline{X \cdot V}(X - X \cdot V)$; and if $p \in U \cdot T^{-1}(z)$ is chosen so that $p \in \bar{U}_0$, then since $U_0 = U \cdot \bar{R}$ we must have $p \in U_0$ and hence $z \in X \cdot V(\overline{X - X \cdot V})$. But since $z \in X$ and $X \cdot T(E) = 0$, it follows that T^* is interior at p. Hence z must be interior to $T^*(U_0) = V$, contrary to $z \in \overline{X - X \cdot V}$. Thus $X \subset V$. Now since every point of $W \cdot T(E)$ is a limit point of X (and thus also of V) it follows that

$$\bar{V} = T(\bar{U}_0) \supset W.$$

Whence

$$T[V_\epsilon(x) \cdot \bar{R}] \supset T(\bar{U}_0) \supset W.$$

Thus $T(x)$ is interior to the image of arbitrarily small open subsets of $\bar{R}$ containing x and accordingly $T^*(\bar{R}) = B_0$ is interior at x, contrary to $x \in E$.

(1.51). COROLLARY. *If a point x of E is an isolated point of E, $T(x)$ is a local separating point of B_0.*

2. THEOREM. *If A is a 2-dimensional manifold (with or without boundary curves) and $T(A) = B$ is interior and light, then B is a 2-dimensional manifold.*

For the purpose of this proof a point x will be called a *regular point* of A or B provided there exists a neighborhood U of x (in A or B) such that $\bar{U}$ is

a 2-cell whose boundary curve does not contain x. Any non-regular point will be called *singular*.

The proof will be given in the form of a series of statements.

(i) *The boundary of any component of the complement of a compact locally connected set in B is locally connected.*

To prove this, let R be a component of $B - M$ where M is a compact locally connected set in B and let $F = F(R)$. Let $T^{-1}(M) = N$, let S be a component of $T^{-1}(R)$ and let $E = F(S)$. Then [2] N is locally connected. Also S is a component of $A - N$. Accordingly E is locally connected. Now since [2] $T(S) = R$ it follows that $T(E) = F$. Therefore F is locally connected, because T is continuous.

(ii) *For any $b \in B$ and any $\epsilon > 0$ there exists a simple closed curve or an arc X in B which irreducibly separates B into just two components and so that the component of $B - X$ containing b is of diameter $< \epsilon$.*

Proof of (ii). Let $a \in B - b$, let $b' \in T^{-1}(b)$ and let U be a 2-cell neighborhood of b' with boundary curve J such that the topological boundary $F(U)$ of U is either J or an arc *cvd* of J and such that $\bar{U} \cdot T^{-1}(b) = b'$, $\bar{U} \cdot T^{-1}(a) = 0$. [Note $T^{-1}(b)$ is finite by (1.2).] Now since, by (1.21), B has no local separating point, there exists a locally connected subcontinuum M of B which ϵ-separates b in B and is such that $T^{-1}(M) \cdot U \neq 0$, $T^{-1}(M) \cdot F(U) = 0$, and such that the component S^* of $B - M$ containing a also contains $T[F(U)]$. Let R be the component of $B - \bar{S}^*$ containing b, let S be the component of $B - \bar{R}$ containing b and let X be the common boundary of R and S. Then X is a compact locally connected subset of M [by (i)] which irreducibly separates a and b in B and $\delta(R) < \epsilon$. Now if R' and S' denote the components of $T^{-1}(R)$ and $T^{-1}(S)$ containing b' and $F(U)$ respectively and if $Z = U \cdot T^{-1}(X)$, it follows that since $b' = U \cdot T^{-1}(b)$ and $T^{-1}(a) \cdot \bar{U} = 0$ there can be no other components of either $T^{-1}(R)$ or $T^{-1}(S)$ intersecting $\bar{U}$. Accordingly $Z = F(R')$ and $Z \subset F(S')$. Hence Z is a continuum and since it is locally connected (because X is locally connected) it must be either a simple closed curve or a simple arc according as b' is a regular point or a singular point of A. Furthermore, since $Z = F(R')$ and $T(R') = R$ it follows that $T(Z) = X$, and since the transformation $T(Z) = X$ is interior on Z, X is either an arc or a simple closed curve.[2] Finally, since there can exist no component of $A - T^{-1}(X)$ other than R' and S' with boundary points in Z, clearly $R + S = B - X$ so that X irreducibly separates B.

Now continuing with the same notation we next prove

(iii) *If X is a simple closed curve, $R + X = \bar{R}$ is a 2-cell with boundary curve X; and thus b is a regular point.*

For if X is a simple closed curve, Z must be a simple closed curve and $R' + Z = \bar{R}'$ is a 2-cell with boundary curve Z. Furthermore the transformation $T(\bar{R}') = \bar{R}$ is interior, since $\bar{R}' = U \cdot T^{-1}(\bar{R})$. Thus since $Z = \bar{R} \cdot T^{-1}(X)$ it follows by (1.4) that $\bar{R}$ is a 2-cell with boundary curve X.

(iii′) *Every singular point of A maps into a singular point of B.*

This is a corollary to (iii).

Still retaining the notation of (ii) we next establish

(iv) *A point x of X is a singular point of B if and only if it is an end point of X.*

First let x be a non-end-point of X and let $q \in T^{-1}(x)$. Let σ be arbitrarily small. Since x is not a cut point of B there exists a subcontinuum N of B such that $B - x \supset N \supset B - V_{\sigma/2}(x)$ and $N \supset T(u) + T(v)$, where u and v are points of z so chosen that the arc uqv of z maps topologically [2] into an arc exf of X under T. Now we can construct in A an arbitrarily small simple closed curve $C = rhs + rks$ such that rhs and rks are arcs in $R' + r + s$ and $S' + r + s$ respectively, $r + s = C \cdot Z$, and $T(C) \cdot N = 0$. Now since C separates x and u in A it follows that $T(C)$ must separate $T(x)$ and N in B. Hence by the argument used in proving (ii), $T(C)$ must contain a set X' irreducibly separating x and N in B and X' is either an arc or a simple closed curve. But now since the set $T^{-1}(X')$ must separate x and u in A and T is topological on uqv, it follows that $T^{-1}(X')$ contains continua H and K such that $r + s \subset H \subset R' + r + s$, $r + s \subset K \subset S' + r + s$. Accordingly X' contains the continua $T(H)$ and $T(K)$. Also $T(H) \cdot T(K) = T(r) + T(s)$ and $T(r) \neq T(s)$. Hence X' cannot be an arc and thus it is a simple closed curve. Therefore, by (iii), x is a regular point.

Now if x is an end point of X, since clearly x can have no 2-cell neighborhood not having x on its boundary curve it follows that x is a singular point.

Now let K denote the set of all singular points of B. Clearly K is closed. It may be vacuous in case A has no singular points, but not otherwise, by (iii′).

(v) *Every point of K is of order 2 (rel. K).*

Proof. By (iv) it follows that every point of K is of order ≤ 2. Now if there were a point x of K of order < 2, there would exist an arbitrarily small locally connected continuum X in B such that $X \cdot K$ contains < 2

points and X irreducibly ϵ-separates x in B. By (iii) it follows that X is an arc and by (iv) that both its end points must belong to K, contrary to supposition.

(v′) *Each component of K is a simple closed curve.*

(vi) $B - K$ *is connected.*

If this were not so we could find a subset K' of K and two distinct components R and S of $B - K'$ with a boundary point p in common, $p \epsilon K'$. But by (ii) and (iv) there exists an arbitrarily small arc ab in B with $ab \cdot K = a + b$ and so that ab separates p from both r and s where $r \epsilon R$, $s \epsilon S$. Clearly this is impossible, since we would then have $ab - (a + b) \subset R \cdot S$.

(vii) *K has only a finite number of components. In fact K has at most p components,*[9] *where $p - 1$ is the maximum number of disjoint simple closed curves in A whose sum does not separate A.*

To prove this, we note first by (vi) it follows that there are only a finite number of components of $A - T^{-1}(K)$. Thus since each component of $T^{-1}(K)$ contains [2] a simple closed curve and since A is disconnected by the removal of any set of p disjoint simple closed curves it follows that the number of components of $T^{-1}(K)$—and hence also of K—must be finite.

To show that the number of components of K does not exceed p, let us suppose the contrary. Let F_1 be the sum of a collection of p components of $T^{-1}(K)$ and let R_1 and S_1 be components of $A - F_1$. If R_1 contains a component of $T^{-1}(K)$, let F_2 be the sum of a set of p-components of $T^{-1}(K)$ selected so that $F_2 \subset \bar{R}_1$ and F_2 conatins at least one component of $T^{-1}(K)$ lying in R_1. Let R_2 and S_2 be components of $A - F_2$ where $S_2 \supset S_1$. Then $R_2 \subset R_1$. Similarly if R_2 contains a component of $T^{-1}(K)$ we select a set F_3 of p components of $T^{-1}(K)$ so that $F_3 \subset \bar{R}_2$ and F_3 contains at least one component of $T^{-1}(K)$ lying in R_2 and take components R_3 and S_3 of $A - F_3$ so that $S_3 \supset S_2$ and hence $R_3 \subset R_2$. Continuing this process, since the number of components of $T^{-1}(K)$ is finite, we eventually find a component R_k of $A - F_k$ such that $R_k \cdot T^{-1}(K) = 0$. But then R_k is a component of $A - T^{-1}(K)$. Hence [2] $T(R_k) = B - K$ and thus $T(F_k) = K$. Clearly this is impossible if K contains more than p components, since F_k has exactly p components.

(viii) *Every singular point of B has a 2-cell neighborhood.*

Proof. Let $p \epsilon K_i \subset K$ where K is the set of all singular points of B and

[9] Actually, $p \leq p_2^1(A) + a + 1$, where a is the number of boundary curves of A and $p_2^1(A)$ is the Betti number, modulo 2, of A.

K_i is a component of K. Let u and v be points of $K_i - p$, let q be a point of $T^{-1}(p)$. Since $T^{-1}(K)$ is a finite graph, there exists a 2-cell neighborhood U of q with boundary J (which may or may not contain q) such that $\bar{U} \cdot T^{-1}(K)$ consists of a finite number of arcs $qx_0, qx_1, \cdots, qx_n$ such that (a) $qx_i \cdot qx_j = q$, (b) $x_0, x_1, \cdots, x_n$ are cyclicly ordered on J and the arc x_0x_n of J not containing q is the set theoretic boundary of U in case q is on J, (c) qx_0 maps topologically into a subarc either of pu or pv, say pu, qx_1 maps topologically onto a subarc of pv, qx_2 into a subarc of pu, and so on to qx_n, where pu and pv are the arcs of K_i not containing v and u respectively, (d) $\bar{U} \cdot T^{-1}(p) = q$.

Now let ab be an arc in B irreducibly ϵ-separating p in B where $a \epsilon pu$, $b \epsilon pv$ and where ϵ is so chosen that the component R of $B - ab$ containing p satisfies $\bar{R} \cdot T(\bar{U} - U) = 0$. Let S be the other component of $B - ab$. Since $T^{-1}(S) \supset x_0x_1$ (arc of J) and $T^{-1}(R) \supset q$, it follows that the open 2-cell V in U bounded by the simple closed curve $G = qx_0 + x_0x_1 + qx_1$ contains a component R' of the inverse of the connected region $Q = R - \widehat{apb}$ (open arc of K_i). Since $x_0x_1 \subset T^{-1}(S)$, it follows at once that the boundary of R' is a simple closed curve $W = a'q + qb' + a'b'$, where $a'q$ and qb' are arcs on qx_0 and qx_1 respectively and where $a'b'$ is an arc which maps into ab and lies except for a' and b' in V. Since $a'b'$ is the common boundary of R' and the component S' of $T^{-1}(S)$ containing x_0x_1 and $a' = T^{-1}(ab) \cdot qx_0$, $b' = T^{-1}(ab) \cdot qx_1$, it follows that $a'b' = T^{-1}(ab) \cdot \bar{V}$ and hence that $a'b'$ maps interiorly onto ab. Thus since $T^{-1}(a) \cdot a'b' = a'$, $T^{-1}(b) \cdot a'b' = b'$ it follows that $a'b'$ maps topologically onto ab.

Therefore we have shown that the boundary W of the open 2-cell R' maps topologically into the boundary C of the region Q while R' maps onto Q. Hence surely the transformation $T(\bar{R}') = \bar{Q}$ is interior, and since $T^{-1}(C) \cdot \bar{Q} = W$, it follows by (1.4) that $\bar{Q}$ is a 2-cell with boundary C.

Clearly the statements (i)–(viii) yield our theorem.

3. Theorem. *If A is a sphere and $T(A) = B$ is interior and light then B is either a 2-cell, a sphere or a projective plane*[11] *(a 2-cell if B has singular points, otherwise a sphere or projective plane).*

Proof. If B has no singular point then by § 2 B is a closed 2-dimensional manifold without boundary; and since by a theorem of the author's[10] $p^1(B)$

[10] See my paper "On the mapping of Betti groups under interior transformations," *Duke Mathematical Journal*, vol. 4 (March, 1938). Here $p^1(X)$ denotes the Betti number (modulo 0) of a set X.

[11] That each of these image sets is possible is shown incidentally in the proof of (3.1).

$\leq p^1(A) = 0$, it follows that B is either a sphere or a projective plane according as it is or is not orientable.

If B has singular points, then by § 2, (vii), the set K of such singular points is a simple closed curve. Let R be a component of $A - T^{-1}(K)$ and let C be the boundary of R, where R is so chosen that $A - \bar{R}$ is connected. (There are only a finite number of components of $A - T^{-1}(K)$ since by § 2, (vi), $B - K$ is connected.) Then since [2] $T^{-1}(K)$ is locally connected it follows that C is a simple closed curve and $\bar{R}$ is a 2-cell. Furthermore $T(\bar{R}) = B$ and $T^{-1}(K) \cdot \bar{R} = C$; and since no subset of $T(C) = K$ can locally separate B at any one of its points it follows by (1. 5) that the transformation $T(\bar{R}) = B$ is interior. Therefore, by (1. 4), B is a 2-cell.

(3. 1). Corollary. *Let $T(A) = B$ be interior and light. If A is a 2-cell, so also is B; if A is a sphere or projective plane, B is either a 2-cell, sphere or projective plane.*

Proof. First let A be the 2-cell $x^2 + y^2 \leq r^2$ in the (x, y) plane. Then the transformation $T'(x, y, z) = (x, y, 0)$ on the sphere A': $x^2 + y^2 + z^2 = r^2$ is interior and light and maps A' into A. Hence TT' is interior and light and maps A' into B; and since by § 2, (iii′), B necessarily has singular points, it follows from the above theorem that B is a 2-cell.

Next let A be a projective plane. The transformation T' obtained on a sphere A' by identifying diametrically opposite points is interior and light and maps A' into a projective plane which we may suppose is A. Then TT' is interior and light and maps A' onto B and hence again our result follows from the theorem just proved.

Finally, the case of the sphere is identical with the above theorem.

4. Applications to other surfaces.

(4. 1). Theorem. *Let $T(M) = N$ be continuous and light where M is a locally connected continuum and suppose that for each $y \in N$, $T^{-1}(y)$ disconnects no cyclic element of M. Then for each A-set [12] N_a in N, each component of $T^{-1}(N_a)$ is an A-set in M.*

Proof. Let K be a component of $T^{-1}(N_a)$. Let R be any component of $M - K$. Now $F(R)$ must reduce to a single point. For if not, then since any two points of $F(R)$ are conjugate,[12] there exists a cyclic element E of M con-

[12] An A-set in a locally connected continuum M is a closed subset K of M which contains every arc in M whose endpoints lie in K. See Kuratowski and Whyburn, *Fundamenta Mathematicae*, Vol. 16 (1930), pp. 305-331.

taining $F(R)$. Since $R \cdot E$ is non-vacuous and connected, it follows that $R \cdot E$ is not a subset of $T^{-1}(N_a)$. Let S be a component of $E - E \cdot T^{-1}(N_a)$. Then $T(S)$ is a connected subset of $N - N_a$ and hence $F(S)$ must map into the single point p which is the boundary of the component of $N - N_a$ containing $T(S)$. Therefore $F(S)$ is totally disconnected and is $\subset T^{-1}(p)$; and since $K \cdot E$ is connected and non-degenerate, $T^{-1}(p)$ must disconnect E, contrary to hypothesis. Hence $F(R)$ is a single point and thus[12] K is an A-set.

(4.11). Corollary. *If $T(M) = N$ is continuous and light where M is a locally connected continuum such that*

(a) *M is unicoherent or*

(b) *every true cyclic element is a cantorian manifold of dimension ≥ 2 then each component of the inverse of an A-set in N is an A-set in M.*

(4.12). Corollary. *If $T(M) = N$ is interior and light and for no $y \in N$ does $T^{-1}(y)$ disconnect any cyclic element in M, then for each A-set N_a in N, $T^{-1}(N_a)$ is the sum of a finite number of disjoint A-sets in M, each of which maps interiorly onto N_a.*

(4.2). Theorem. *If $T(A) = B$ is interior and light where A is a locally connected continuum which either*

(a) *is unicoherent, or*

(b) *has as its true cyclic elements sets which locally are cantorian manifolds of dimension ≥ 2,*

then for each true cyclic element E_b in B there exists a true cyclic element E_a in A which maps interiorly onto E_b under T.

Proof. By the preceding theorem and corollaries, $T^{-1}(E_b)$ is made up of a finite number of disjoint A-sets each mapping onto E_b. Let E_a be a node[13] of any one of these components K of $T^{-1}(E_b)$. Then since the transformation $T(K) = E_b$ is interior it follows[10] that E_a maps onto E_b under T. Furthermore, the transformation $T(E_a) = E_b$ is interior except possibly at the one point $p = E_a \cdot \overline{K - E_a}$. If this transformation failed to be interior at p, there would exist a point $q \in E_a - p$ such that $T(q) = T(p)$. But then since $T(E_a) = E_b$ is interior at q it follows by (1.21) that $T(p)$ is not a local separating point of E_b; and hence by (1.51), $T(E_a) = E_b$ is interior at p. Thus $T(E_a) = E_b$ is interior and our theorem is proven.

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[13] A node of a locally connected continuum M is either an end point of M or a true cyclic element of M containing exactly one cut point of M. See my paper in the *American Journal of Mathematics*, vol. 50 (1928), pp. 167-194.

This theorem, together with the results established in §§ 2, 3, yield the following corollaries.

(4.21). *Under the conditions of the theorem, B likewise satisfies* (a) *or* (b) *respectively.*

(4.22). *If every true cyclic element of a locally connected continuum A is a 2-dimensional manifold (with or without boundary curves) the same is true of any light interior image of A.*

(4.23). *If every true cyclic element of a locally connected continuum is either a sphere, 2-cell or projective plane, the same is true of any light interior image of A.*

Now if we make use of the fact [2] that any interior transformation T can be factored into the form $T = T_2T_1$ where T_1 is monotone and T_2 is interior and light we get:

(4.24). *If every true cyclic element of a locally connected continuum A is either a sphere or a 2-cell and if $T(A) = B$ is any interior transformation (light or not), then each true cyclic element of B is either a sphere, 2-cell, or projective plane.*

For if we factor T into T_2T_1 where $T_1(A) = A'$ is monotone and $T_2(A') = B$ is interior and light, it follows [14] that every true cyclic element of A' is either a sphere or a 2-cell. Hence by (4.23) we get our conclusion.

(4.3). THEOREM. *Let $T(A) = B$ be interior and light where A is a locally connected continuum and every true cyclic element of A is locally a cantorian manifold of dimension $\geqq 2$. Then for each true cyclic element E_b in B we have*

$$T^{-1}(E_b) = \sum_{i=1}^{k} E_a{}^i$$

where for each i, $E_a{}^i$ is a true cyclic element of A and the transformation $T(E_a{}^i) = E_b$ is interior.

Proof. Let K be any component of $T^{-1}(E_b)$. By (4.1), K is an A-set in A. Since [10] each node of K maps onto all of E_b it follows that there are only a finite number of nodes of K. Hence if E_a is any true cyclic element in

[14] See R. L. Moore, *Transactions of the American Mathematical Society*, vol. 27 (1925), pp. 416-428; C. B. Morrey, *American Journal of Mathematics*, vol. 57 (1935), pp. 17-50; G. T. Whyburn, *ibid.*, vol. 56 (1934), pp. 294-302.

K, then E_a contains just a finite set F of cut points of K, i. e., the set $F = E_a \cdot \overline{K - E_a}$ is finite. Hence $T(F) = H$ is finite. Now, since, by (4. 21), E_b is a cantorian manifold of dimension ≥ 2, it follows that $E_b - T(F)$ is connected; and since by hypothesis E_a is such a manifold, $E_a - E_a T^{-1}(H) = G$ is likewise connected. But G is a component of $K - K \cdot T^{-1}(H)$, since $T^{-1}(H) \supset F$. Hence [2] G maps onto all of $E_b - T(F)$ and thus E_a maps onto all of E_b under T. Since T is interior on $E_a - F$ and F is finite, and since E_b has no local separating point, it follows by (1. 5) that T is interior on E_a.

Since each true cyclic element of K maps interiorly onto all of E_b under T, it follows that there are only a finite number of such elements in K. And since clearly K can have no arc of cut points [for $T(K) = E_b$ is interior], it follows that K is the sum of a finite number of true cyclic elements each mapping interiorly onto E_b. Thus since $T^{-1}(E_b)$ has only a finite number of components, our theorem follows.

(4. 31). Corollary. *If every true cyclic element of A is a 2-dimensional manifold and $T(A) = B$ is interior and light, for each non-end-point $b \in B$, $T^{-1}(b)$ is a finite set.*

If b belongs to a true cyclic element of B, this follows from (4. 3) and (1. 2). If b is a cut point of B it results at once from the easily established fact that $T^{-1}(b)$ is contained in the sum of a finite number of cyclic elements of A.

(4. 32). Corollary. *If every true cyclic element of A is a sphere or 2-cell and $T(A) = B$ is interior, then for each non-end-point $b \in B$, $T^{-1}(b)$ has only a finite number of components.*

This follows from (4. 31) using the factorization of T into a monotone transformation and a light interior transformation just as was done in the proof of (4. 24).

It may be remarked that it follows from the above results that any one dimensional interior image of a sphere—or of any locally connected continuum every true cyclic element of which is a sphere, 2-cell, or projective plane—is necessarily a dendrite. Also it is interesting to note that—by sending certain indecomposable continua into points—a sphere may be mapped interiorly into *any dendrite*. However, if we specify that our interior transformation shall send only locally connected sets into single points, it results that the only possible one-dimensional images of a 2-dimensional manifold are the *arc* and the *simple closed curve*. A detailed study of these and closely related results will be made in a later paper.

5. In conclusion, we shall isolate—essentially from the properties developed in § 2—a theorem which yields a complete analysis "in the small" of any light interior transformation on a 2-dimensional manifold. We begin by establishing

(5.1). THEOREM. *Let $T(A) = B$ be interior and light. Suppose E is a 2-cell with interior R and generator J in A such that $E \cdot T^{-1}T(J) = J$ and T is $(1-1)$ on J. Then T is $(1-1)$ on E, so that the transformation $T(E) = F$ is topological.*

Proof. Suppose, on the contrary, that there exist two points x and y in R such that $T(x) = T(y) = p$. Let apb be an arc in F such that $apb \cdot T(J) = a + b$. There exist arcs $a'xb'$ and $a'yb'$ in E which map topologically onto apb under T, where $a' = E \cdot T^{-1}(a)$, $b' = E \cdot T^{-1}(b)$. Since these two arcs are different, there exists a component S of $E - (a'xb' + a'yb')$ such that $S \cdot J = 0$ and hence so that $S \subset R$. Clearly this is impossible, since S contains a component U of $A - T^{-1}(apb)$ and U necessarily maps onto one of the components of $F - apb$ and each of these contains points of $T(J)$.

(5.2). THEOREM. *Let $T(A) = B$ be interior and light, where A is a 2-dimensional manifold. For any point q of A there exists a closed 2-cell neighborhood E of q in A and a positive integer k such that* (a) *if $T(q)$ is a regular point of B, then on E T is topologically equivalent to the transformation $w = z^k$ on $|z| \leq 1$;* (b) *if $T(q)$ is a singular point of B, then on E T is topologically equivalent to the transformation*

$$f(z) = \rho(\cos k\theta/2 + i \mid \sin k\theta/2 \mid) \qquad (k \text{ even})$$

on $|z| \leqq 1$ when q is a regular point of A and to this same transformation (for a different value of k) on $|z| \leq 1$, $y \geq 0$ when q is a singular point.

Proof. Let us consider first the case where $T(q)$ is a singular point of B. Then, referring back to the proof of (viii) in § 2, it follows by (5.1) that the set $\bar{R}'$ there defined maps topologically under T onto the set $\bar{Q}$. In that proof let us set $V = V_0$, $a' = a_0$, $b' = a_1$, $R' = R_0$. Let V_i be the open 2-cell in U bounded by the simple closed curve $qx_i + x_ix_{i+1} + qx_{i+1}$ for $i < n$ if q is on J and for $i \leqq n$ $(n + 1 \equiv 0)$ if q is within J. Then in exactly the same way it follows that there exist arcs $a_1a_2, a_2a_3, \cdots, a_ia_{i+1}, \cdots$ in $\bar{V}_1, \bar{V}_2, \cdots, \bar{V}_i, \cdots$ so that if R_i is the open 2-cell in V_i bounded by the simple closed curve $qa_i + a_ia_{i+1} + qa_{i+1}$ (where $0 \leq i \leq n - 1$ for q on J and $0 \leq i \leq n$, $n + 1 \equiv 0$, if Q is within J), $\bar{R}_i$ maps topologically onto $\bar{Q}$ under T. Hence

if we call E the closed 2-cell $\Sigma \bar{R}_i$, it follows that, on E, T is topologically equivalent to the transformation $f(z) = \rho(\cos n\theta + i \mid \sin n\theta \mid)$ on $\mid z \mid \leq 1$, $y \geq 0$, if q is on J and to the transformation

$$f(z) = \rho[\cos(n+1)\theta/2 + i \mid \sin(n+1)\theta/2 \mid] \qquad (n+1 \text{ even})$$

on $\mid z \mid \leq 1$ if q is within J. Thus (b) is established.

Now suppose $T(q) = b$ is a regular point of B. Let xby be an arc in B consisting wholly of regular points. Referring back to the proofs of (ii) and (iii) in § 2, we see that since $T^{-1}(xby)$ is a finite graph we can choose the simple closed curve X in those proofs so that

$$T^{-1}(xby) \cdot (R' + Z) = qx_1 + qx_2 + \cdots + qx_{2k}$$

where (1) $q = b'$, (2) the points $x_1, x_2, x_3, \cdots, x_{2k}$ are cyclically ordered on Z, (3) $qx_1, qx_2, \cdots$ are simple arcs intersecting in only q and such that qx_1 maps topologically onto the arc bu of xby, qx_2 onto bv, qx_3 onto bu, and so on to qx_{2k} which maps topologically onto bv, where $xby \cdot \bar{R}$ is the arc ubv, (4) the arc x_1x_2 of Z maps topologically onto one of the arcs, say urv, of X from u to v, x_2x_3 maps topologically onto the other arc, say usv, of Z, x_3x_4 maps onto urv, x_4x_5 onto usv and so on to $x_{2k}x_1$ which maps topologically onto usv. Thus by (5.1) it follows that the closed 2-cells $\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_{2k}$ in $\bar{R}'$ bounded by the simple closed curves $qx_i + x_ix_{i+1} + qx_{i+1}$ $(0 \leq i \leq 2k, 2k+1 \equiv 1)$ map topologically and alternately onto the two 2-cells in $\bar{R}$ bounded by the closed curves $ubv + urv$ and $ubv + usv$. Hence, on $\bar{R}'$, T is topologically equivalent to the transformation $w = z^k$ on $\mid z \mid \leq 1$.

UNIVERSITY OF VIRGINIA.

ON THE DISTRIBUTION FUNCTIONS OF ALMOST PERIODIC FUNCTIONS.*

By Philip Hartman, E. R. van Kampen and Aurel Wintner.

Introduction. While it is known [1] that every almost periodic function $z(t), -\infty < t < +\infty$, has an asymptotic distribution function σ, very little is known about sufficient conditions which, when imposed on $z(t)$, insure a preassigned degree of smoothness for the function σ. Everything that is known about the subject indicates that this problem of smoothness depends essentially on at least two factors, namely

(i) the smoothness properties of the given function $z(t)$;

(ii) the arithmetical structure of the sequence of Fourier exponents of $z(t)$.

Actually, it is not (i) that is needed so much but rather

(i bis) the local smoothness properties of the function Z which is a continuous function of the position on a finite or infinite dimensional torus which is associated with $z(t)$ in the usual way.[2]

In fact, a condition of the type (i) is weaker than a condition of the type (i bis). For Z is constructed from the given function $z(t)$ by using the limiting process of the Kronecker-Weyl approximation theorem; and nothing seems to be known [3] about which, if any, of the smoothness properties of z (e. g., differentiability of a given degree or analyticity) are transplanted by this process into corresponding, or somewhat weaker, smoothness properties of the function Z on the torus. As far as the factor (ii) is concerned, it is known [4] that the situation for the smoothness of σ is most favorable in case that the Fourier exponents of $z(t)$ are linearly independent or such that $z(t)$ is of the form

$$z(t) = \Sigma z_\nu(\lambda_\nu t), \tag{1}$$

where the $z_\nu(t)$ are continuous functions of a fixed period T and the λ_ν are linearly independent. Correspondingly, it is to be expected that the chances

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[1] Wintner [10], [11]; Haviland [3]. For a comprehensive account of a general theory, cf. Jessen and Wintner [6].

[2] Bohr [1].

[3] Cf. Wintner [13].

[4] Wintner [12]; Jessen and Wintner [6]; Kershner and Wintner [8]; van Kampen and Wintner [7].

for the smoothness of σ are least favorable [5] in case $z(t)$ is limit periodic, i. e., of the form

$$z(t) = \sum_{\nu=1}^{\infty} z_\nu(r_\nu t), \tag{2}$$

where the $z_\nu(t) \not\equiv z_0(0)$ are continuous functions of some common period and the r_ν are positive rational numbers such that lim inf $r_\nu = 0$.

The object of the present paper is to fill somewhat the gap between these two extreme cases. The method to be applied will be an extension of the one recently [6] applied in the most favorable case (1). From the point of view of the factor (ii), the smoothness problem of σ in case of an arbitrary trigonometric polynomial, i. e., of a finite number of Fourier exponents, is hardly different from the case of any almost periodic function whose Fourier exponents are generated by a sequence of linearly independent numbers (or more, generally, moduli). Correspondingly, when proving smoothness properties of σ, it will be a methodically unimportant simplification to assume that Z is a function on a finite dimensional torus.[7]

The results imply, in particular, that if $z(t)$ is a trigonometric polynomial, then its distribution function σ is absolutely continuous on the spectrum of σ. In particular, any non-constant trigonometric polynomial $x + iy = z(t)$ maps the t-axis on a connected set whose closure is either a finite set of analytic arcs or the closure of a two-dimensional open set which is bordered by a finite number of analytic arcs; and σ is necessarily absolutely continuous with a density which is always an analytic, though not necessarily regular analytic, function of the position on the spectrum, and which need not be one and the same analytic function on the entire spectrum. While this case of a trigonometric polynomial seems to be quite harmless, it is, as a fact, hardly easier than the general case to be considered.

As far as the factor (i bis) is concerned, the condition imposed on the function Z of the position on the finite dimensional torus is that either Z is regular analytic or that Z has continuous partial derivatives of order k, where $1 \leqq k \leqq \infty$. In the second case, the results are similar to, although from the geometrical point of view more complicated than, those mentioned for the case of a trigonometric polynomial.

1. The open set *R*. Let $Z = Z(\theta_1, \cdots, \theta_n)$ be a continuous function of the position on the torus

[5] Cf. Bohr [2], where, however, the terms of the series (2) are not free of constant stretches.

[6] van Kampen and Wintner [7].

[7] As to a theory on an infinite dimensional torus, cf., Jessen [4], [5].

(3) $\quad \Theta: \quad 0 \leqq \theta_j < 1; \; j = 1, \cdots, n, \qquad (1 \leqq n < +\infty),$

which is obtained from the n-dimensional Cartesian θ-space by reduction to modulus 1; and let T denote the corresponding mapping of Θ on the complex plane, so that

(4) $\quad T: \quad x + iy \equiv z = Z(\theta_1, \cdots, \theta_n).$

By $T(\Lambda)$ will be denoted the image of a subset Λ of Θ under the continuous transformation T. While the continuous mapping T of Θ on $T(\Theta)$ is not, in general, topological, the set of all those points of Θ whose T-image is in a set F will be denoted by $T^{-1}(F)$, so that a subset Λ of Θ may be a proper subset of $T^{-1}(T(\Lambda))$. By meas Λ and $\mu(F)$ will be denoted the ordinary n-dimensional Lebesgue measure on Θ and the 2-dimensional Lebesgue measure in the (x, y)-plane, respectively.

It will be assumed that

(I) the function (4) is of class C_k, $k \geqq 1$, where C_k for $k \geqq 1$ is the class of functions for which all continuous partial derivatives of order k exist (and C_0 is the class of continuous functions).

(II) the mapping (4) has the property that meas $T^{-1}(P) = 0$ for every point P of the (x, y)-plane.

Condition (II) excludes the case that Z is constant on an open subset of Θ. Even if the function (4) has partial derivatives of arbitrarily high order and is nowhere constant, it is quite possible that condition (II) is not satisfied; for $T^{-1}(P)$ can be, for particular points P, a nowhere dense perfect set of positive measure.

Let $\lambda_1, \cdots, \lambda_n$ be n linearly independent real numbers, and $z(t)$ the almost periodic function

(5) $\quad x(t) + iy(t) \equiv z(t) = Z(\lambda_1 t, \cdots, \lambda_n t), \quad (-\infty < t < +\infty).$

Let E denote a Borel set in the (x, y)-plane, and $\sigma(E)$ the asymptotic distribution function of $z(t)$. It is clear from the Kronecker-Weyl approximation theorem that

(6) $\quad \sigma(E) = \text{meas}\, T^{-1}(E).$

Obviously, condition (II) is equivalent to the assumption that $\sigma(E)$ has no point spectrum, i. e., that $\sigma(E) = 0$ whenever E consists of a single point. Since T is continuous, it is clear from (6) that the spectrum of $\sigma(E)$ is the set $T(\Theta)$.

Let Ω denote the set of those points of Θ at which the matrix

$$\begin{pmatrix} X_{\theta_1} & \cdots & X_{\theta_n} \\ Y_{\theta_1} & \cdots & Y_{\theta_n} \end{pmatrix}, \tag{7}$$

formed by the partial derivatives of the real and imaginary parts of (4), is not of rank 2; so that Ω consists of the zeros $(\theta_1, \cdots, \theta_n)$ of the Jacobian square sum

$$\Delta \equiv \Delta(\theta_1, \cdots, \theta_n) = \sum_{1 \leqq j < l \leqq n} \begin{vmatrix} X_{\theta_j} & X_{\theta_l} \\ Y_{\theta_j} & Y_{\theta_l} \end{vmatrix}^2. \tag{8}$$

Since the function (8) is continuous on Θ, the set Ω is closed. Let R denote the complement,

$$R = T(\Theta) - T(\Omega), \tag{9}$$

of the T-image of Ω with respect to the spectrum $T(\Theta)$, so that R is contained in, but is not necessarily identical with, $T(\Theta - \Omega)$.

It is easy to see that the set R, which may be empty, is always open. In fact, if P_0: (x_0, y_0) is a point of $T(\Theta - \Omega)$, then $P_0 = T(\Pi_0)$ for at least one point Π_0: $(\theta_1^0, \cdots, \theta_n^0)$ of the subset $\Theta - \Omega$ of Θ. Hence, the function (8) does not vanish at $(\theta_1^0, \cdots, \theta_n^0)$; in other words, at least one of the two-rowed Jacobian determinants occurring in (8), say the one for which $j = 1$, $l = 2$, does not vanish at $(\theta_1^0, \cdots, \theta_n^0)$. Thus, a small vicinity of the point $(\theta_1, \theta_2) = (\theta_1^0, \theta_2^0)$ of the (θ_1, θ_2)-subspace $\theta_3 = \theta_3^0, \cdots, \theta_n = \theta_n^0$ of Θ is mapped by T on a vicinity of P_0: (x_0, y_0) in a topological way, so that $T(\Theta - \Omega)$ is an open set. Obviously, $T(\Omega)$ is a closed set. Hence, R is an open set in the (x, y)-plane.

2. The distribution function on R. It will be shown that if R is not empty, the completely additive set function $\sigma(E)$ is absolutely continuous on R and has there a density of class C_{k-1}, i. e., there exists on R a (non-negative) function $\delta = \delta(x, y)$ of class C_{k-1} such that

$$\sigma(E) = \int_E \delta(x, y)\, dxdy \text{ for every } E \subset R. \tag{10}$$

First, if P_0: (x_0, y_0) is any point of R, every point of the closed set $T^{-1}(P_0)$ is contained in the open set $\Theta - \Omega$ on which the rank of the matrix (7) is 2. Hence, $T^{-1}(P_0)$ is not only a closed set, but it is also an $(n-2)$-dimensional manifold, with the property that the common part of $T^{-1}(P_0)$ and of any sufficiently small sphere contained in Θ is a connected set. Since Θ is compact, it follows that $T^{-1}(P_0)$ consists of a finite number, say $m = m(P_0)$, of mutually disjoint, connected closed $(n - 2)$-dimensional manifolds

$\mathbf{N}_1, \cdots, \mathbf{N}_m$. Since P_0 is any point of the open set R, the compactness of Θ also implies the existence of a sufficiently small $\epsilon > 0$ such that the closure $\bar{C}$ of the circle C defined by

$$C: \qquad PP_0^2 = (x - x_0)^2 + (y - y_0)^2 < \epsilon^2$$

is contained in R and $m(P) = m(P_0)$ holds for every point $P: (x, y)$ of the closed set $\bar{C}$. The T^{-1}-image of C consists of m mutually disjoint, connected open sets on Θ. If $\Lambda_1, \cdots, \Lambda_m$ denote these open sets, then every Λ_j is a T^{-1}-image of C and every point of $T^{-1}(C)$ is contained in $\Lambda_1 + \cdots + \Lambda_m$. It will be supposed that the enumeration has been chosen in such a way that $\Lambda_q \supset \mathbf{N}_q$, $q = 1, \cdots, m$.

It is clear from the definition of $\mathbf{N}_q$ that there exists in Θ a finite number of spheres, say p_q, such that these p_q spheres cover $\mathbf{N}_q$ and the common part Γ_{qr}^{n-2} of $\mathbf{N}_q$ and the r-th sphere can be parametrized by functions of class C_k defined on an $(n-2)$-dimensional sphere.

Suppose now that $k \geqq 2$ in the assumption (I) of section 2 and that [8] the dimension number $n \geqq 3$. Then if $\Pi_0: (\theta_1^0, \cdots, \theta_n^0)$ is any point on $\mathbf{N}_q$ and if $\mathbf{M}_0 = \mathbf{M}(\theta_1^0, \cdots, \theta_n^0)$ denotes the (2-dimensional) plane normal to $\mathbf{N}_q$ at Π_0, then there exists an $\eta > 0$ (independent of Π_0) such that the η-vicinity of Π_0 on $\mathbf{M}_0$ does not contain points of any other normal plane $\mathbf{M}(\theta_1, \cdots, \theta_n)$ for any point $\Pi: (\theta_1, \cdots, \theta_n) \neq \Pi_0$ on any $\mathbf{N}_j$, $j = 1, \cdots, m$. Furthermore, this vicinity of Π on $\mathbf{M}$ is the topological map of a neighborhood of $P_0: (x_0, y_0)$ under the transformation T; and this topological correspondence is given by functions of class C_k.

Let $\epsilon > 0$ be so small that the T^{-1}-image of the circle C, namely $\Lambda_1, \cdots, \Lambda_m$, is such that all points of the open set Λ_q are within a distance η of the set $\mathbf{N}_q$. It is clear from the above considerations that Λ_q may be considered to be a "cylindrical tube" obtained by taking the image of C on all normal planes $\mathbf{M}(\theta_1, \cdots, \theta_n)$ as the point $(\theta_1, \cdots, \theta_n)$ varies over $\mathbf{N}_q$. Let Γ_{qr}^n $(q = 1, \cdots, m;\ r = 1, \cdots, p_q)$ denote the n-dimensional set consisting of images of C on all normal planes $\mathbf{M}(\theta_1, \cdots, \theta_n)$ as $(\theta_1, \cdots, \theta_n)$ varies over Γ_{qr}^{n-2}.

Now there exist functions of class C_k,

$$\theta_j = \vartheta_j^{qr}(x, y, \alpha_1, \cdots, \alpha_{n-2}), \tag{11}$$

$(j = 1, \cdots, n;\ q = 1, \cdots, m;\ r = 1, \cdots, p_q)$, defined on the product space

[8] Cf. E. R. van Kampen and A. Wintner, [7], pp. 181-182 for a case analogous to $n = 2$.

of the $(n-2)$-dimensional sphere $\mathrm{A}: \sum_{j=1}^{n-2} \alpha^2{}_j < 1$ and the set C on the (x, y)-plane, so that (11) is a parametrization of $\Gamma_{qr}{}^n$ for fixed q, r. In fact, for fixed q, r and fixed $x = x_0$, $y = y_0$, the functions (11) denote a parametrization of $\Gamma_{qr}{}^{n-2}$ (the same, of course, is true if (x_0, y_0) is replaced by an arbitrary point of C and $\Gamma_{qr}{}^{n-2}$ by the corresponding set of Θ); also, these functions (11) give, for fixed q, r and fixed $(\alpha_1, \cdots, \alpha_{n-2})$, a parametrization of the map of the circle C on the corresponding normal plane $\mathbf{M}$. The Jacobian

$$J^{qr}(x, y, \alpha_1, \cdots, \alpha_{n-2}) = \frac{\partial(\vartheta_1{}^{qr}, \cdots, \vartheta_n{}^{qr})}{\partial(x, y, \alpha_1, \cdots, \alpha_{n-2})}$$

does not vanish on the product space of A and C.

It is clear from the above analysis of the mapping T^{-1} in a neighborhood of P_0, that (10) is satisfied for every open E (hence for every Borel set) in the ϵ-vicinity of P_0, if one puts

$$(12) \quad \delta(x, y) = \sum_{q=1}^{m} \sum_{r=1}^{p_q} \int \cdots \int | J^{qr}(x, y, \alpha_1, \cdots, \alpha_{n-2}) | \, d\alpha_1 \cdots d\alpha_{n-2}$$

where the integral of $| J^{qr} |$ extends over that part of the sphere A which corresponds to points of $\Gamma_{qr}{}^n$ not contained in $\Gamma_{qj}{}^n$, $j = 1, \cdots, r-1$. This function (12) is of class C_{k-1}. Since P_0 is an arbitrary point of R, the proof is complete for $k \geqq 2$.

Obvious modifications of this proof assure the result for the case that the function (4) is of class C_1.

3. A criterion for the absolute continuity of $\sigma(E)$. If one does not introduce conditions in addition to those assumed so far, one cannot state that the asymptotic distribution function (6) of (5) is absolutely continuous, i. e., that there exists a (non-negative) measurable function $\delta(x, y)$ such that

$$(13) \qquad \sigma(E) = \int\int_E \delta(x, y)\, dx dy$$

holds not only for every Borel set E contained in the open (and possibly empty) set R but for every Borel set E contained in the spectrum of σ. Since $T(\Theta)$ is the spectrum, it is clear that the distribution function σ is absolutely continuous if

$$(14) \qquad \sigma(T(\Omega)) = 0.$$

It will be shown that if Ω is a zero set on Θ, i. e., if

$$(15) \qquad \operatorname{meas} \Omega = 0,$$

then $\sigma(E)$ is an absolutely continuous distribution function and that, in addition, the measure $\mu(R)$ of the open subset (9) of the spectrum $T(\Theta)$ is identical with $\mu(T(\Theta))$. Since (10) was shown to hold for a continuous function $\delta(x, y)$ on R, it follows that if E is any closed rectangle in the (x, y)-plane, then the Lebesgue integral (13) is a proper or possibly improper Riemann integral according as E does not or does contain points of the boundary $T(\Omega)$ of the open set R.

First, it is clear from (6) that the statement (14) is equivalent to

$$\text{meas}\, T^{-1}(T(\Omega)) = 0, \tag{16}$$

where $T^{-1}(T(\Omega))$ is not to be confused with Ω. On the other hand, the assumption (15) implies, for every $\epsilon > 0$, the existence of an open set Γ_ϵ in Θ such that $\Omega \subset \Gamma_\epsilon$ and meas $\Gamma_\epsilon < \epsilon$. Hence, in order to prove (16), it is sufficient to show that the common part of $T^{-1}(T(\Omega))$ and $\Theta - \Gamma_\epsilon$ is a zero set on Θ. Since the set $\Theta - \Gamma_\epsilon$ is closed and is contained in the open set $\Theta - \Omega$ on which the function (8) does not vanish, it follows by arguments similar to those used in section 2 that it is sufficient to prove the relation $\mu(T(\Omega)) = 0$. But this relation is obvious from (8) and the definition of Ω.

This proves that (15) implies (14) and also that $\mu(R) = \mu(T(\Theta))$, the last relation being, in view of (9), equivalent to $\mu(T(\Omega)) = 0$.

It may be mentioned, that instead of Lebesgue measure, one could have used Jordan content, since the zero sets involved are closed sets.

4. The analytic case. It is seen from the proof of (10), (11) and (12) that if (4) is a regular analytic function of the n real variables $\theta_1, \cdots, \theta_n$ on Θ, then $\delta(x, y)$ is a regular analytic function of the position on the open set R of the real (x, y)-plane (it is understood that R need not be connected). Furthermore, it is clear from the definition of Ω that, in the analytic case, Ω consists of a finite number of manifolds each if which has a dimension number less than the dimension number n of Θ, unless the function (8) vanishes identically on Θ. It follows, therefore, from section 3 that in the analytic case either (8) vanishes identically on Θ or the distribution function $\sigma(E)$ is absolutely continuous with a density $\delta(x, y)$ which is a regular analytic function of the real variables (x, y) on G_i, where $G_1, G_2, \cdots$ is a sequence of mutually disjoint connected open sets in the real (x, y)-plane and $R = \Sigma G_i$. This implies, in particular, that if Z is regular analytic, $\Delta \not\equiv 0$, and $\lambda_1, \cdots, \lambda_n$ are linearly independent, then the closure of the set of values z attained by the almost periodic function (5) for $-\infty < t < +\infty$ is the closure of an open set in the z-plane, i. e., such as to have no one-dimensional parts. For this

16

geometrical restriction on the spectrum of σ is a necessary condition for the absolute continuity of σ.

The case of a trigonometric polynomial $z(t)$, as described in the introduction, affords, among other things, the simplification that one can always find a finite number of linearly independent exponents $\lambda_1, \cdots, \lambda_n$ and a regular analytic function Z on Θ such that (5) is satisfied.

It may be mentioned that if it is only known that the nowhere constant function (4) has continuous partial derivatives of arbitrarily high order, it is quite possible that the distribution function is not absolutely continuous, although the spectrum is two-dimensional or even a Jordan region. This holds even in the particular case (1) of convolutions.[9]

5. The case $\Delta \equiv 0$. There remains to be considered the case of a regular analytic mapping (4) such that $\Delta \equiv 0$ on Θ (a case illustrated by a real-valued Z). There are two cases possible according as the matrix (7) is or is not of rank 0 at every point of Θ. In the first case, the function (4) is a constant on Θ. In the second case, the rank of (7) is 1 at all points of Θ which do not belong to a set Ω_0 consisting of a finite number of manifolds each of which has a dimension number less than the dimension number n of Θ. In what follows, only the latter case will be considered.

Since all the two-rowed Jacobians formed by the elements of the matrix (7) vanish at all points of Θ, while at least one of the $2n$ partial derivatives occurring in (7) does not vanish at every point of $\Theta - \Omega_0$, it is easy to show that the image of the torus Θ under the analytic mapping (4) is a one-dimensional connected analytic manifold of finite arc length, or, more precisely, the spectrum $T(\Theta)$ of $\sigma(E)$ consists of a sequence of analytic arcs which can have singularities and which have a total finite arc length. In order to see this, no use need to be made of known general theorems concerning analytic mappings of the compact set Θ (theorems which apply also to the case considered in the previous section).

Since the spectrum of σ is one-dimensional, hence a zero set in the (x, y)-plane, the set function $\sigma(E)$ cannot be absolutely continuous in the sense of (13). Correspondingly, it is plausible to replace the definition (13) of absolute continuity by the requirement that if s is the local length parameter on $T(\Theta)$, then

$$\sigma(S) = \int_S \delta(s)\, ds \tag{13 bis}$$

for every open arc S (or for every Borel set S) on $T(\Theta)$ and for a suitable

[9] In this connection, cf. Kershner [9].

(non-negative) function $\delta(s)$ of the position s on $T(\Theta)$. Then an obvious adaptation of the considerations of sections 2, 3 shows that the distribution function $\sigma(E)$ is an absolutely continuous on $T(\Theta)$ in this sense, and that $T(\Theta)$ contains a sequence of mutually disjoint open arcs $A_1, A_2, \cdots$ such that $\delta(s)$ is regular analytic on every A_i and $\tau(T(\Theta) - \Sigma A_i) = 0$, where $\tau(S)$ denotes the s-measure defined by

$$\tau(S) = \int_S ds.$$

Actually, there are only a finite number of A_i and, correspondingly, only a finite number of G_i in the previous section.

6. Dependence of the distribution function on the moduli. Assuming only that the function (4) is continuous on Θ, denoting by λ the vector formed by the n real numbers $(\lambda_1, \cdots, \lambda_n)$, and by $\sigma_\lambda(E)$ the distribution function of the almost periodic function (5), the distribution function (6), which is independent of λ, is identical with $\sigma_\lambda(E)$ whenever the n components of λ are linearly independent. On the other hand, $\sigma_\lambda(E)$ depends on λ in a rather unstable way, if one does not require that the n components of λ be linearly independent. Nevertheless, the instability referred to does not appear at those points of the λ-space at which the n components of λ are linearly independent.

In order to formulate this statement in a precise manner, let L_ν, where $\nu = 0, 1, \cdots, n$, denote the set of those points of the n-dimensional vector space of λ at which there exist exactly ν dependencies between the n components of λ, these dependencies being homogeneous, linear, and having integral coefficients. Thus, L_n consists of the single point which represents the origin of the λ-space, while L_0 is a set of points λ for which $\sigma_\lambda(E)$ is given by (6). The complement, $L^* = L_1 + \cdots + L_n$, of L_0 is everywhere dense in the λ-space, but is contained in a dense sequence of hyperplanes. If the function (4) of n variables is not a continuous function of the position on a torus whose dimension number is less than n, it is easy to see that the distribution function $\sigma_\lambda(E)$, considered as a functional on the λ-space, $L_0 + L^*$, is discontinuous at every point λ of L^*.

Now, the statement is that the functional of λ is continuous at every point of L_0, although the set L^* of the discontinuity points is everywhere dense in the λ-space. In other words, $\sigma_\lambda \to \sigma_{\lambda_0}$ when λ tends on $L_0 + L^*$ to an arbitrary point λ_0 on L_0. Since $\sigma_\lambda(E)$ is independent of λ on L_0, it is sufficient to prove that $\sigma_\lambda \to \sigma_{\lambda_0}$ when λ tends on L^* to an arbitrary point λ_0 on L_0. But this follows by inspection of the existence proof (cf., Wintner [11]; Haviland [3]) of the asymptotic distribution function σ_λ of the almost periodic function (5).

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THE FOURIER COEFFICIENTS OF THE MODULAR INVARIANT $J(\tau)$.*

By Hans Rademacher.

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1. Recently Dr. Zuckerman and I have developed general formulae for the Fourier coefficients of modular forms of positive dimensions.[1] We remarked at the end of our paper that the series obtained would be convergent also for forms of dimension zero, i. e. for *modular functions*, among which $J(\tau)$ can be regarded as fundamental. The question arises whether the formally constructed series for the coefficients of $J(\tau)$ actually represents them.

The solution of this problem requires a thorough revision of our method, since we had essentially made use of the positivity of the dimension of the modular forms. A method due to Kloosterman [2] and later extended by Estermann [3] gives the clue. Kloostermann's method consists of two devices: first it improves the estimate of certain sums $A_k(n)$ of roots of unity from the trivial one

$$|A_k(n)| \leqq k$$

to

$$(1.1) \qquad |A_k(n)| \leqq Ck^{1-\beta+\epsilon} \cdot (k, n)^{\beta},$$

where β, according to results of Salié [4] and Davenport [5] can be taken as $\beta = \frac{1}{3}$. Secondly it collects the Farey arcs $\xi_{h,k}$ belonging to the same k and treats the resulting sum as a whole instead of estimating the summands separately. Both of these expedients will be used in the following paper.

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[1] "On the Fourier coefficients of certain modular forms of positive dimension," to be published in the *Annals of Mathematics*.

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[3] T. Estermann, "Vereinfachter Beweis eines Satzes von Kloosterman," *ibid.*, vol. 7 (1929), pp. 82-98.

[4] H. Salié, "Zur Abschätzung der Fourierkoeffizienten ganzer Modulformen," *Mathematische Zeitschrift*, vol. 36 (1933), pp. 263-278.

[5] H. Davenport, "On certain exponential sums," *Journal f. d. reine u. angew. Mathematik*, vol. 169 (1933), pp. 158-176.

2. Our problem is to investigate the coefficients of the expansion

$$12^3 J(\tau) = e^{-2\pi i\tau} + \sum_{n=0}^{\infty} c_n e^{2\pi i n\tau} = f(e^{2\pi i\tau}), \tag{2.1}$$

where $J(\tau)$ is defined by means of the modular forms $g_2(\omega_1, \omega_2)$ and $g_3(\omega_1, \omega_2)$ as

$$J(\tau) = \frac{g_2{}^3(1,\tau)}{g_2{}^3(1,\tau) - 27 g_3{}^2(1,\tau)} = \frac{g_2{}^3(1,\tau)}{\Delta(1,\tau)}, \qquad \Im(\tau) > 0. \tag{2.2}$$

A consequence of (2.2) is, by the way, the formula [6]

$$12^3 J(\tau) = \frac{\left\{ 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 x^m}{1 - x^m} \right\}^3}{x \left\{ \sum_{\lambda=-\infty}^{+\infty} (-1)^\lambda x^{\frac{\lambda(3\lambda-1)}{2}} \right\}^{24}}, \qquad x = e^{2\pi i\tau}, \tag{2.3}$$

which shows that the coefficients c_n in (2.1) are integers. From (2.2) we infer the invariance of $J(\tau)$ with respect to the transformations of the full modular group:

$$J\left(\frac{a\tau + b}{c\tau + d}\right) = J(\tau). \tag{2.4}$$

We shall see that the equations (2.1) and (2.4) will completely suffice for the determination of the coefficients c_n, provided $n \geqq 1$. For

$$\tau = \frac{iz}{k} + \frac{h}{k}, \qquad \Re(z) > 0,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix}$$

with

$$hh' \equiv -1 \pmod{k}, \tag{2.5}$$

equation (2.4) goes over into

$$J\left(\frac{iz}{k} + \frac{h}{k}\right) = J\left(\frac{i}{kz} + \frac{h'}{k}\right)$$

or, in the notation of (2.1),

$$f\left(e^{-\frac{2\pi z}{k} + \frac{2\pi i h}{k}}\right) = f\left(e^{-\frac{2\pi}{kz} + \frac{2\pi i h'}{k}}\right). \tag{2.6}$$

3. From (2.1) we obtain

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(x)}{x^{n+1}}\, dx = \sum_{\substack{h,k \\ 0 \leqq h < k \leqq N}}' \frac{1}{2\pi i} \int_{\xi_{h,k}} \frac{f(x)}{x^{n+1}}\, dx,$$ [7]

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[6] Klein–Fricke, *Vorlesungen über die Theorie der Modulfunktionen*, vol. I, p. 154.

[7] Σ' means here and subsequently that h runs over integers prime to k.

where the $\xi_{h,k}$ may be the Farey arcs of order N of the circle C

$$|x| = e^{-2\pi N^{-2}}.$$

If we introduce on $\xi_{h,k}$ the new variable ϕ through

$$x = \exp(-2\pi N^{-2} + \frac{2\pi i h}{k} + 2\pi i\phi)$$

we get

$$c_n = \sum_{\substack{h,k \\ 0 \leqq h < k \leqq N}}' e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f(e^{\frac{2\pi i h}{k} - 2\pi(N^{-2} - i\phi)}) e^{2\pi n(N^{-2} - i\phi)} d\phi. \tag{3.1}$$

For a later purpose we need here the determination of $\vartheta'_{h,k}$ and $\vartheta''_{h,k}$ in terms of h and k. In the Farey series of order N we consider the fraction h/k with its two neighbors:

$$\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}, \qquad k, k_1, k_2 \leqq N. \tag{3.2}$$

We have here

$$hk_1 - h_1 k = 1, \qquad h_2 k - hk_2 = 1,$$

and therefore

$$hk_1 \equiv 1 \pmod{k}, \qquad hk_2 \equiv -1 \pmod{k}$$

or, from (2.5),

$$k_1 \equiv -h', \qquad k_2 \equiv h' \pmod{k}. \tag{3.3}$$

The Farey segment around h/k is bounded by the mediants between the fractions (3.2)

$$\frac{h_1 + h}{k_1 + k}, \qquad \frac{h_2 + h}{k_2 + k}.$$

Since these mediants do not belong to the Farey series of order N we have

$$k_1 + k > N, \qquad k_2 + k > N,$$

which conditions, together with (3.2), enclose k_1 and k_2 in the intervals

$$N - k < k_1 \leqq N, \qquad N - k < k_2 \leqq N. \tag{3.4}$$

The formulae (3.3) and (3.4) determine k_1 and k_2 uniquely as functions of h and k. In particular we have

$$\vartheta'_{h,k} = \frac{1}{k(k_1 + k)}, \qquad \vartheta''_{h,k} = \frac{1}{k(k_2 + k)}. \tag{3.5}$$

4. In (3.1) we apply the transformation formula (2.6) and obtain

$$(4.1)\qquad c_n = \sum_{\substack{h,k \\ 0 \leqq h < k \leqq N}}' e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f\left(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{k^2 w}}\right) e^{2\pi n w}\, d\phi$$

with the abbreviation

$$(4.2)\qquad w = N^{-2} - i\phi.$$

If we now write

$$(4.3)\qquad \begin{aligned} f(x) &= x^{-1} + D(x), \\ D(x) &= \sum_{m=0}^{\infty} c_m x^m \end{aligned}$$

we can accordingly split the expression (4.1) into two parts:

$$(4.4)\qquad c_n = Q(n) + R(n)$$

with

$$(4.41)\qquad Q(n) = \sum_{\substack{h,k \\ 0 \leqq h < k \leqq N}}' e^{-\frac{2\pi i}{k}(nh+h')} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi}{k^2 w} + 2\pi n w}\, d\phi,$$

$$(4.42)\qquad R(n) = \sum_{\substack{h,k \\ 0 \leqq h < k \leqq N}}' e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} D\left(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{k^2 w}}\right) e^{2\pi n w}\, d\phi.$$

In $Q(n)$, which we consider first, we divide the intervals of integration into three parts according to

$$-\vartheta'_{h,k} = -\frac{1}{k(k_1+k)} \leqq -\frac{1}{k(N+k)} < \frac{1}{k(N+k)} \leqq \frac{1}{k(k_2+k)} = \vartheta''_{h,k}$$

and get

$$(4.5)\qquad \begin{aligned} Q(n) &= \sum_{k=1}^{N} \sum_{h \bmod k}' e^{-\frac{2\pi i}{k}(nh+h')} \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} \\ &\quad + \sum_{k=1}^{N} \sum_{h \bmod k}' e^{-\frac{2\pi i}{k}(nh+h')} \int_{-\frac{1}{k(k_1+k)}}^{-\frac{1}{k(N+k)}} \\ &\quad + \sum_{k=1}^{N} \sum_{h \bmod k}' e^{-\frac{2\pi i}{k}(nh+h')} \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(k_2+k)}} \\ &= Q_0(n) + Q_1(n) + Q_2(n), \end{aligned}$$

say. The integrand in all three integrals of (4.5) is the same as in (4.41).

5. In $Q_0(n)$ we can immediately perform the summation with respect to h since the integral is independent of h. Setting

$$A_k(n) = \sum_{h \bmod k}' e^{-\frac{2\pi i}{k}(nh+h')} \tag{5.1}$$

we get

$$Q_0(n) = \sum_{k=1}^{N} A_k(n) \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} e^{\frac{2\pi}{k^2 w}+2\pi n w}\, d\phi$$

or

$$Q_0(n) = \sum_{k=1}^{N} A_k(n) \frac{1}{i} \int_{N^{-2}-\frac{i}{k(N+k)}}^{N^{-2}+\frac{i}{k(N+k)}} e^{\frac{2\pi}{k^2 w}+2\pi n w}\, dw, \tag{5.2}$$

where we have introduced w from (4.2) as variable of integration. We remark further that $A_k(n)$ is a Kloosterman sum (cf. the references in § 1) and can therefore be estimated as

$$|A_k(n)| < Ck^{2/3+\epsilon} \cdot (k,n)^{1/3}, \tag{5.3}$$

where (k,n) is the greatest common divisor of k and n. In the complex w-plane we consider now the closed rectangular path R with the four vertices

$$\pm N^{-2} \pm \frac{i}{k(N+k)}.$$

We take R as surrounding 0 in the positive sense. Then we have

$$\begin{aligned}
Q_0(n) &= 2\pi \sum_{k=1}^{N} A_k(n) \frac{1}{2\pi i} \int_R e^{\frac{2\pi}{k^2 w}+2\pi n w}\, dw \\
&\quad - \frac{1}{i} \sum_{k=1}^{N} A_k(n) \left\{ \int_{N^{-2}+\frac{i}{k(N+k)}}^{-N^{-2}+\frac{i}{k(N+k)}} + \int_{-N^{-2}+\frac{i}{k(N+k)}}^{-N^{-2}-\frac{i}{k(N+k)}} + \int_{-N^{-2}-\frac{i}{k(N+k)}}^{N^{-2}-\frac{i}{k(N+k)}} \right\} \\
&= 2\pi \sum_{k=1}^{N} A_k(n) L_k(n) - \frac{1}{i} \sum_{k=1}^{N} A_k(n) \{J_1 + J_2 + J_3\},
\end{aligned} \tag{5.4}$$

say, where all four integrals have the same integrand.

For an estimation of J_1 and J_3 we observe that on their paths of integration we have

$$w = u \pm \frac{i}{k(N+k)},$$
$$-N^{-2} \leqq u \leqq N^{-2},$$
$$\Re(w) = u \leqq N^{-2},$$
$$\Re\left(\frac{1}{w}\right) = \frac{u}{u^2 + \dfrac{1}{k^2(N+k)^2}} < N^{-2}k^2(N+k)^2 \leqq 4k^2,$$

so that

$$\left| e^{\frac{2\pi}{k^2 w} + 2\pi n w} \right| \leqq e^{8\pi + 2\pi n N^{-2}}$$

and therefore

$$\begin{matrix} |J_1| \\ |J_3| \end{matrix} \leqq 2N^{-2} e^{8\pi + 2\pi n N^{-2}}. \tag{5.5}$$

In J_2 we have

$$w = -N^{-2} + iv,$$
$$-\frac{1}{k(N+k)} \leqq v \leqq \frac{1}{k(N+k)},$$
$$\Re(w) = -N^{-2} < 0, \quad \Re\left(\frac{1}{w}\right) = \frac{-N^{-2}}{N^{-4} + v^2} < 0,$$

hence

$$\left| e^{\frac{2\pi}{k^2 w} + 2\pi n w} \right| < 1$$

and therefore

$$|J_2| < \frac{2}{k(N+k)} < 2k^{-1}N^{-1}. \tag{5.6}$$

Combining (5.3), (5.5), and (5.6) we obtain

$$\sum_{k=1}^{N} A_k(n)\{J_1 + J_2 + J_3\} = O\left(e^{2\pi n N^{-2}} \sum_{k=1}^{N} k^{2/3+\epsilon}(n,k)^{1/3} k^{-1} N^{-1}\right)$$

and for $n \geqq 1$, which we assume from now on, we have $(n,k) \leqq n$ and hence

$$\sum_{k=1}^{N} A_k(n)\{J_1 + J_2 + J_3\} = O\left(e^{2\pi n N^{-2}} n^{1/3} N^{-1/3+\epsilon}\right). \tag{5.7}$$

Furthermore we have

$$L_k(n) = \frac{1}{2\pi i}\int_R e^{\frac{2\pi}{k^2 w}+2\pi n w}\,dw$$

$$= \frac{1}{2\pi i}\int_R \sum_{\mu=0}^{\infty}\frac{\left(\frac{2\pi}{k^2 w}\right)^{\mu}}{\mu\,!}\sum_{\nu=0}^{\infty}\frac{(2\pi n w)^{\nu}}{\nu\,!}\,dw$$

$$= \frac{1}{k\sqrt{n}}\sum_{\nu=0}^{\infty}\frac{\left(\frac{2\pi\sqrt{n}}{k}\right)^{2\nu+1}}{\nu\,!\,(\nu+1)\,!}$$

or

$$L_k(n) = \frac{1}{k\sqrt{n}}\,I_1\!\left(\frac{4\pi\sqrt{n}}{k}\right), \tag{5.8}$$

where $I_1(z)$ is the Bessel function of first order with purely imaginary argument. From (5.4), (5.7), (5.8) we deduce

$$Q_0(n) = \frac{2\pi}{\sqrt{n}}\sum_{k=1}^{N}\frac{A_k(n)}{k}\,I_1\!\left(\frac{4\pi\sqrt{n}}{k}\right) + O(e^{2\pi n N^{-2}}n^{1/3}N^{-1/3+\epsilon}). \tag{5.9}$$

6. We now turn our attention to $Q_1(n)$ and $Q_2(n)$ in (4.5), of which we discuss only $Q_2(n)$ in detail since $Q_1(n)$ can be treated in quite the same manner. We have from (4.5)

$$Q_2(n) = \sum_{k=1}^{N}\sum_{h \bmod k}{}' e^{-\frac{2\pi i}{k}(nh+h')}\int_{\frac{1}{k(N+k)}}^{\frac{1}{k(k_2+k)}} e^{\frac{2\pi}{k^2 w}+2\pi n w}\,d\phi$$

$$= \sum_{k=1}^{N}\sum_{h \bmod k}{}' e^{-\frac{2\pi i}{k}(nh+h')}\sum_{l=k_2+k}^{N+k-1}\int_{\frac{1}{k(l+1)}}^{\frac{1}{kl}} e^{\frac{2\pi}{k^2 w}+2\pi n w}\,d\phi,$$

$$Q_2(n) = \sum_{k=1}^{N}\sum_{l=N+1}^{N+k-1}\int_{\frac{1}{k(l+1)}}^{\frac{1}{kl}} e^{\frac{2\pi}{k^2 w}+2\pi n w}\,d\phi \sum_{\substack{h \bmod k\\ N<k_2+k\leq l}}{}' e^{-\frac{2\pi i}{k}(nh+h')}. \tag{6.1}$$

In the inner sum of the last expression the restriction imposed on k_2 means, in consequence of (3.3), a restriction of h' to an interval modulo k, which is equivalent to one interval or to two intervals in the range $0 \leqq h' < k$. Therefore the sum in question is an incomplete Kloosterman sum, for which we have the estimate [8]

[8] This estimate of the incomplete sum does not seem to appear explicitly in the

$$(6.2)\qquad \sum_{\substack{h \bmod k\\ N-k<k_2\leqq l-k}}{}' e^{-\frac{2\pi i}{k}(nh+h')} = O(k^{2/3+\epsilon}(n,k)^{1/3}) = O(k^{2/3+\epsilon}n^{1/3}).$$

In the integral in (6.1) we have

$$\begin{aligned}\Re\left(\frac{2\pi}{k^2w}+2\pi nw\right) &= \Re\left(\frac{2\pi}{k^2(N^{-2}-i\phi)}+2\pi n(N^{-2}-i\phi)\right)\\ &= 2\pi\left(\frac{N^{-2}}{k^2(N^{-4}+\phi^2)}+nN^{-2}\right) \leqq 2\pi\left(\frac{N^{-2}}{k^2N^{-4}+\dfrac{1}{(k+N)^2}}+nN^{-2}\right)\\ &< 2\pi\left(\left(\frac{k+N}{N}\right)^2+nN^{-2}\right) \leqq 8\pi+2\pi nN^{-2},\end{aligned}$$

and from this and (6.2) we get

$$\begin{aligned}Q_2(n) &= O\left(e^{2\pi nN^{-2}}n^{1/3}\sum_{k=1}^{N}\sum_{l=N+1}^{N+k-1}\left(\frac{1}{kl}-\frac{1}{k(l+1)}\right)k^{2/3+\epsilon}\right)\\ &= O\left(e^{2\pi nN^{-2}}n^{1/3}\sum_{k=1}^{N}\frac{1}{k^{1/3-\epsilon}N}\right),\end{aligned}$$

$$(6.3)\qquad Q_2(n) = O(e^{2\pi nN^{-2}}n^{1/3}N^{-1/3+\epsilon}).$$

Since a similar result is valid for $Q_1(n)$ we derive from (4.5), (5.9), and (6.3):

$$(6.4)\qquad Q(n) = \frac{2\pi}{\sqrt{n}}\sum_{k=1}^{N}\frac{A_k(n)}{k}I_1\left(\frac{4\pi\sqrt{n}}{k}\right)+O(e^{2\pi nN^{-2}}n^{1/3}N^{-1/3+\epsilon}).$$

7. From (4.42) and (4.3) we obtain

$$R(n) = \sum_{k=1}^{N}\sum_{h \bmod k}{}' e^{-\frac{2\pi inh}{k}}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}}\sum_{m=0}^{\infty}c_m e^{\frac{2\pi ih'm}{k}-\frac{2\pi m}{k^2w}}e^{2\pi nw}\,d\phi.$$

We decompose again the Farey segment $-\vartheta'_{h,k}\leqq\phi\leqq\vartheta''_{h,k}$ in the Kloosterman manner and have, after an interchange of the summations with respect to h and m,

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literature. Incomplete sums for general k are given in Kloosterman's paper with a less precise estimate, and Davenport treats only the case k equal to a prime number, but with the precision (6.2). By the device, however, which Estermann uses *loc. cit.*, p. 94, or by the other one, which Davenport applies *loc. cit.*, pp. 173, 174, we can reduce the estimation of the incomplete sum to that of the complete sum. Thus we obtain (6.2) from Salié's estimate, *loc. cit.*, p. 264. We are for our purpose not particularly interested in the lowest possible value of the exponent of k in (6.2), as long as it is a constant less than 1.

$$(7.1)\quad R(n)=\sum_{k=1}^{N}\sum_{m=0}^{\infty}c_m\int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}}e^{-\frac{2\pi m}{k^2w}+2\pi nw}d\phi\sum_{h\bmod k}{}' e^{-\frac{2\pi i}{k}(nh-mh')}$$

$$+\sum_{k=1}^{N}\sum_{m=0}^{\infty}c_m\sum_{h\bmod k}{}' e^{-\frac{2\pi i}{k}(nh-mh')}\sum_{l=k_1+k}^{N+k-1}\int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}}e^{-\frac{2\pi m}{k^2w}+2\pi nw}d\phi$$

$$+\sum_{k=1}^{N}\sum_{m=0}^{\infty}c_m\sum_{h\bmod k}{}' e^{-\frac{2\pi i}{k}(nh-mh')}\sum_{l=k_2+k}^{N+k-1}\int_{\frac{1}{k(l+1)}}^{\frac{1}{kl}}e^{-\frac{2\pi m}{k^2w}+2\pi nw}d\phi$$

$$=S_1+S_2+S_3.$$

In all integrals of (7.1) we have

$$(7.2)\quad \Re\left(\frac{2\pi m}{k^2w}\right)=\frac{2\pi mN^{-2}}{k^2(N^{-4}+\phi^2)}\geqq\frac{2\pi m}{k^2N^{-2}+N^2k^2\vartheta^2_{h,k}}\geqq\frac{2\pi m}{1+1}=\pi m.$$

The complete Kloosterman sum in S_1 admits of an estimate

$$\sum_{h\bmod k}{}' e^{-\frac{2\pi i}{k}(nh-mh')}=O(k^{2/3+\epsilon}(n,k)^{1/3})=O(k^{2/3+\epsilon}n^{1/3})$$

which holds uniformly in m. We obtain therefore

$$S_1=O\left(\sum_{k=1}^{N}\sum_{m=0}^{\infty}|\,c_m\,|\,\frac{2}{kN}e^{-\pi m+2\pi nN^{-2}}k^{2/3+\epsilon}n^{1/3}\right)$$

$$=O\left(e^{2\pi nN^{-2}}n^{1/3}N^{-1}\sum_{m=0}^{\infty}|\,c_m\,|\,e^{-\pi m}\sum_{k=1}^{N}k^{-1/3+\epsilon}\right),$$

$$(7.3)\quad S_1=O(e^{2\pi nN^{-2}}n^{1/3}N^{-1/3+\epsilon}).$$

The sums S_2 and S_3 are both of the same structure so that we need to treat only one of them. By interchanging the summations with respect to h and l we get

$$S_2=\sum_{k=1}^{N}\sum_{m=0}^{\infty}c_m\sum_{l=N+1}^{N+k-1}\int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}}e^{-\frac{2\pi m}{k^2w}+2\pi nw}d\phi\sum_{\substack{h\bmod k\\ N<k+k_1\leqq l}}{}' e^{-\frac{2\pi i}{k}(nh-mh')}.$$

The inner sum is an incomplete Kloosterman sum, for which we have, uniformly in m, the estimate

$$\sum_{\substack{h \bmod k \\ N-k<k_1\leqq l-k}}' e^{-\frac{2\pi i}{k}(nh-mh')} = O(k^{2/3+\epsilon}(n,k)^{1/3}) = O(k^{2/3+\epsilon}n^{1/3})$$

Therefore, taking note of (7.2), we get

$$S_2 = O\left(\sum_{k=1}^{N}\sum_{m=0}^{\infty} | c_m | \frac{1}{kN} e^{-\pi m+2\pi nN^{-2}} k^{2/3+\epsilon} n^{1/3}\right),$$

$$S_2 = O(e^{2\pi nN^{-2}} n^{1/3} N^{-1/3+\epsilon}). \tag{7.4}$$

From (7.1), (7.3), and (7.4) we infer

$$R(n) = O(e^{2\pi nN^{-2}} n^{1/3} N^{-1/3+\epsilon}) \tag{7.5}$$

and then from (4.4), (6.4), (7.5)

$$c_n = \frac{2\pi}{\sqrt{n}} \sum_{k=1}^{N} \frac{A_k(n)}{k} I_1\left(\frac{4\pi\sqrt{n}}{k}\right) + O(e^{2\pi nN^{-2}} n^{1/3} N^{-1/3+\epsilon}). \tag{7.6}$$

Now we keep here $n > 0$ fixed and let N tend to infinity. The error term then tends to zero. Thus we obtain our main result, which we state in the following

THEOREM. *In the Fourier expansion for the modular function $J(\tau)$*

$$12^3 J(\tau) = e^{-2\pi i t} + c_0 + \sum_{n=1}^{\infty} c_n e^{2\pi i n\tau} \tag{7.71}$$

the coefficients c_n, $n \geqq 1$, are determined by the convergent series

$$c_n = \frac{2\pi}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_1\left(\frac{4\pi\sqrt{n}}{k}\right) \tag{7.72}$$

with

$$A_k(n) = \sum_{h \bmod k}' e^{-\frac{2\pi i}{k}(nh+h')}, \qquad hh' \equiv -1 \pmod{k}. \tag{7.73}$$

8. We had to exclude $n = 0$ in our discussion. This peculiarity is not caused by $\sqrt{n}$ appearing in the denominator in (7.6), as the computation of $L_k(n)$ in the lines preceding (5.8) shows that $n = 0$ is not exceptional in this respect:

$$\left[\frac{1}{k\sqrt{n}} I_1\left(\frac{4\pi\sqrt{n}}{k}\right)\right]_{n=0} = L_k(0) = \frac{2\pi}{k^2}. \tag{8.1}$$

The estimates of the incomplete Kloosterman sums, however, would break down for $n = 0$. By suitable examples it is easy to see that incomplete Kloosterman sums with $n = 0$ do not admit of a better general estimate than $O(k)$, which would, of course, not suffice in our reasonings. The series (7.72), on the other

hand, will remain convergent for $n = 0$, but does not even accidentally represent the coefficient c_0, which can directly be obtained from (2. 3) as

$$c_0 = 744. \tag{8.2}$$

Indeed, we get

$$A_k(0) = \sum_{h \bmod k}' e^{-\frac{2\pi i}{k} h'} = \mu(k)$$

with the Möbius symbol $\mu(k)$, and hence from (8. 1) and (7. 72)

$$2\pi \sum_{k=1}^{\infty} A_k(0) \frac{2\pi}{k^2} = 4\pi^2 \sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} = 4\pi^2 \frac{1}{\zeta(2)} = 4\pi^2 \cdot \frac{6}{\pi^2} = 24,$$

different from (8. 2). There is of course no reason to expect that c_0 might be found by our method, which makes use only of the behavior of $J(\tau)$ at $\tau = i\infty$, expressed by (2. 1), and of the invariance stated in (2. 4). Both properties remain obviously unchanged for any $J(\tau) + C$ instead of $J(\tau)$.

9. The coefficients c_n, which can be found from (2. 3) by troublesome computations, which for higher n are practically inexecutable, do not seem to have attracted much attention before. All I could discover in the literature were, besides c_0, the two coefficients

$$c_1 = 196\,884, \qquad c_2 = 21\,493\,760.$$

Our convergent series (7. 72) gives another approach to the actual computation of the c_n, which, as we know, have to be integers. Unfortunately, the convergence of (7. 72) is rather slow, so that we should need quite a number of terms in order to get an error which is safely less than 1/2. Nevertheless, it is interesting to see that the first few terms of the series already furnish the bulk of the considerable amounts of those coefficients.

We get, for $n = 1$:

$$\begin{aligned}
2\pi \frac{A_1(1)}{1} I_1(4\pi) &= 196\,550.\,665 \\
2\pi \frac{A_2(1)}{2} I_1\left(\frac{4\pi}{2}\right) &= 250.\,822 \\
2\pi \frac{A_3(1)}{3} I_1\left(\frac{4\pi}{3}\right) &= 48.\,535 \\
2\pi \frac{A_4(1)}{4} I_1\left(\frac{4\pi}{4}\right) &= 14.\,110 \\
2\pi \frac{A_5(1)}{5} I_1\left(\frac{4\pi}{5}\right) &= 8.\,380 \\
\sum &= \overline{196\,872.\,512},
\end{aligned}$$

and for $n = 2$:

$$\begin{aligned}
\frac{2\pi}{\sqrt{2}}\ \frac{A_1(2)}{1} I_1(4\pi\sqrt{2}) &= 21\,495\,869.279 \\
\frac{2\pi}{\sqrt{2}}\ \frac{A_2(2)}{2} I_1\left(\frac{4\pi\sqrt{2}}{2}\right) &= -2\,054.739 \\
\frac{2\pi}{\sqrt{2}}\ \frac{A_3(2)}{3} I_1\left(\frac{4\pi\sqrt{2}}{3}\right) &= -\quad 84.640 \\
\frac{2\pi}{\sqrt{2}}\ \frac{A_4(2)}{4} I_1\left(\frac{4\pi\sqrt{2}}{4}\right) &= 0.000 \\
\frac{2\pi}{\sqrt{2}}\ \frac{A_5(2)}{5} I_1\left(\frac{4\pi\sqrt{2}}{5}\right) &= 7.012 \\
\sum &= \overline{21\,493\,736.912.}
\end{aligned}$$

These values are in error only by the comparatively small amounts of -11.488 and -23.088 respectively.

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